

**The contributions of Interactive Dynamic Mathematics software in probing
understanding of mathematical concepts: Case study on the use GeoGebra in
learning the concept of modulus functions.**

A THESIS SUBMITTED TO THE FACULTY OF THE INSTITUTE OF EDUCATION OF
THE UNIVERSITY COLLEGE LONDON BY

Ebert Nhamo Gono

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR IN EDUCATION

Supervisor: Dr. Cosette Crisan

February 2016

© E.N. Gono 2016

ACKNOWLEDGEMENTS

Firstly, I am grateful to God, for giving me the strength to keep working on this thesis.

I am forever thankful to my supervisors Dr Cosette Crisan and Professor Candia Morgan without whose direction, patience, support and constructive criticism, I would not have successfully completed this study.

This study wouldn't have taken off without the six participants who participated in the lessons until the end. I am forever thankful to them.

I am particularly thankful to my children: Mabel; Garikai, Fungai, Darryl and grant children Tasha, Tanisha, Troy and Elinum for being there for me and understanding my quest and how it robbed us of the valuable family time. For those countless times they nagged me on how far I was with my research, applying some indirect pressure on me.

Finally, I would like to express my heartfelt thank you to my extended family scattered all over the world, my friends and the Zimbabwe Catholic Community in England and Wales who carried me through the darkest moments of my life. Special thanks also go to all the friends who proof read my Thesis. May the Lord God bless them all.

DEDICATION

This dissertation is dedicated to my late wife Rhoda. For the love care and support she provided throughout the good and difficult times of our lives, and for encouraging me to live my dream. For all that time I spent away from home pursuing my studies, she tolerated me. May her Soul Rest in Peace.

ABSTRACT:

This study explored how dynamic mathematics software package called GeoGebra, contributed to participants' learning and understanding of mathematics. The research focused on participants' experiences as they used and transformed GeoGebra to support their understanding of modulus functions. It highlighted participants' perspective on the role of GeoGebra in supporting exploration of modulus functions in particular and mathematical ideas in general. The focus of the study was on how participants utilised GeoGebra to address misconceptions and perceptions of modulus functions. The main research questions that guided this study focused on how students used GeoGebra to support their understanding of modulus functions and how GeoGebra related and contributed to their whole learning experiences. The study found that GeoGebra provided a medium for visualisation that linked the abstract aspects of modulus functions with graphical illustrations. The study also noted that working with GeoGebra extended participants' understanding of modulus functions i.e. (a) in distinguishing the difference between the graphs of $y = |f(x)|$ and $y = f(|x|)$ and being able to sketch the two graphs without the aid of GeoGebra; (b) challenging why the integral of $1/x$ is $y = \ln |x| + c$ instead of $y = \ln(x) + c$ since the derivative of $f(x) = \ln(x)$ is $1/x$; (c) recognising that not all turning points are stationary; (d) recognising the importance of graphical illustrations when solving equations and inequalities that involve modulus functions, to eliminate spurious answers derived from some algebraic calculations.

DECLARATION AND WORD LENGTH:

I Ebert Nhamo Gono hereby declare that, except where explicit attribution is made, the work presented in this thesis is entirely my own.

Word count: 43 098

Signature:

TABLE OF CONTENTS

CHAPTER 1:

1.0 Introduction	1
1.1 Background to study	2
1.2 Purpose of the study	5
1.3 Research Questions	6
1.4 Overview of the study.....	9

CHAPTER 2:

2.0 Literature review	12
2.1 Theoretical Framework	13
2.1.1 Philosophical Paradigms	14
2.1.2 The Constructivism approach	15
2.1.3 Instrumental Approach	18
2.1.4 Semiotic Mediation framework	22
2.2 Dynamic Mathematics software	25
2.2.1 Instant feedback	27
2.2.2 Speed of execution	28
2.2.3 Differentiated learning	28
2.3 The role of multiple representations	29
2.4 Case of GeoGebra	33
2.4.1 GeoGebra interface	35
2.4.2 Sliders	39
2.4.3 The uniqueness of GeoGebra	40
2.5 Case of modulus functions	41
2.6 Summary	44

CHAPTER 3	46
3.0 Methodology, design and data collection and analysis	46
3.1 Research Methodology	46
3.2 Research Design	48
3.2.1 The Pilot Study	49
3.2.2 Sampling techniques, sample and part	50
3.2.2.1 Sample	51
3.2.2.2 Participant profiles	52
3.3 Research methods	54
3.4 Methods of Data Collection	54
3.4.1 Screencast Video and Audio recordings	55
3.4.2 Session Observations	57
3.4.3 Interviews	58
3.5 Methods of Data analysis	60
3.5.1 Data transcription	61
3.5.1.1 Viewing videos attentively	62
3.5.1.2 Initial Noting	62
3.5.1.3 Developing emerging themes	62
3.5.1.4 Connections across emergent themes	63
3.5.1.5 Moving to next cases	64
3.5.1.6 Looking for patterns across themes	64
3.6 Trustworthiness of the study	65
3.7 My Role as the researcher	66
3.8 Ethical issues	67
3.9 Summary	70

CHAPTER 4:

4.0 Data collection and data analysis	71
4.1 Session 1: Introduction to GeoGebra	72
4.2 Session 2: Polynomials, exponentials and trig functions	102
4.3 Session 3: Graphs of $y = f(x) $ and $y = f(x)$	113

4.4 Session 4: Solving equations involving modulus functions.....	133
4.5 Session 5: Solving inequalities involving modulus functions	142
4.6 Session 6: Participant responses at the end of the session	144
4.7 Summary	144
CHAPTER 5:	
5.1 Conclusions and discussions	146
5.2 Findings	147
5.3 Discussions	156
CHAPTER 6:	
6.0 Contribution of the study.....	158
6.1 Limitations of the study	160
6.2 Recommendations for further research	160
REFERENCES:	162
APPENDICES:	182
LIST OF TABLES:	
Table 4.1	73
Table 4.2	90
Table 4.3	99
LIST OF SKETCHES:	
Sketch 4.1.....	100
Sketch 4.2	101
LIST OF INSERTS:	
Insert 2.1	42
Insert 2.2	43
Insert 4.1	104

Insert 4.2	133
Insert 4.3	137
Insert 4.4	138
Insert 4.5	138
Insert 4.6	140
Insert 4.7	142
Insert 4.8	143

LIST OF VIGNETTES:

Vignette 1	74
Vignette 2	78
Vignette 3	80
Vignette 4	83
Vignette 5	85
Vignette 6	87
Vignette 7	93
Vignette 8	96
Vignette 9	97
Vignette 10	108
Vignette 11	115
Vignette 12	118
Vignette 13	120
Vignette 14	122

LIST OF FIGS:

Fig 2.1	35
Fig 2.2	36
Fig 2.3	36
Fig 2.4	37
Fig 2.5	39
Fig 4.1	75

Fig 4.2	77
Fig 4.3	79
Fig 4.4	81
Fig 4.5	84
Fig 4.6	84
Fig 4.7	87
Fig 4.8	93
Fig 4.9	94
Fig 4.10	95
Fig 4.11	95
Fig 4.12	97
Fig 4.13	103
Fig 4.14	105
Fig 4.15	106
Fig 4.16	107
Fig 4.17	109
Fig 4.18	111
Fig 4.19	112
Fig 4.20	116
Fig 4.21	117
Fig 4.22	118
Fig 4.23	119
Fig 4.24	120
Fig 4.25	121
Fig 4.26	122
Fig 4.27	124
Fig 4.28	125
Fig 4.29	126
Fig 4.30	127
Fig 4.31	128
Fig 4.32	130

Fig 4.33	132
Fig 4.34	135
Fig 4.35	136
Fig 4.36	137
Fig 4.37	139
Fig 4.38	139
Fig 4.39	141

Edd Reflective statement:

The purpose of this reflective statement is not to give a detailed account of all the courses I have done during my EdD programme, but to highlight aspects of the programme that have impacted on my professional development as a mathematics teacher and researcher. To understand the extent of the impact, I will start by giving a brief background of myself and how I became interested in the area of technology in mathematics teaching and learning. I will discuss some aspects of my assignments that shaped my perception of professionalism, my perception of research and my practice as a mathematics teacher in secondary schools and Further Education in England. These institutions are endowed with vast resources of technology that are rarely used in the teaching and learning of mathematics in particular. Finally I will discuss my journey through my Thesis and how it has impacted on my professional practice.

My background:

I initially trained as a primary school teacher. Joining the teaching service immediately after Zimbabwe gained its independence (1980), I found myself drafted into secondary schools to teach mathematics, to cover for science teacher shortage in an education system that had expanded more than tenfold after the country's independence. I enrolled for a Bachelor in Education degree in Mathematics education, with specific emphasis on teaching 'A' level mathematics, which was a shortage area. In January of 1996, I joined further education as a mathematics lecturer in a Teachers' Training College (an affiliate of the University of Zimbabwe), with added responsibilities of being in-charge of ICT. My own knowledge of computers at that time was limited to personal use, and basic training received during my first degree. I enrolled for a post graduate diploma in computer studies at a local Polytechnic college; I attended a one-month training course in Tel-Aviv (Israel) on how to retrieve information from databases.

The same year (1996) saw five teachers' training colleges in Zimbabwe introduced to the Internet through a United Nations Educational, Scientific and Cultural Organization (UNESCO) sponsored project. UNESCO launched the 'Learning without Frontiers' project which aimed at connecting teachers' training institutions in Zimbabwe through the Internet. Due to my responsibilities as Lecturer in Charge (ICT), I was drafted into

this project as the College's point man. In 2003 I moved to the United Kingdom to join family and moved back to teaching mathematics. I kept an interest in technology and mathematics teaching.

A brief description of and reflection on the content of my assignments:

In October 2008 I enrolled at IOE for the EdD programme. My first module was The Foundations of Professionalism. The Foundation of Professionalism assignment did not focus on the use of technology in mathematics teaching and learning but afforded me an opportunity to reflect upon my own professionalism as a mathematics teacher in England. I chose to explore the impact of the National Curriculum on mathematics teachers' sense of professionalism. My main area of interest was to examine the perception that the National Curriculum framework embodied a shift towards accountability and prescriptive instructional policies. Critics of accountability and prescriptive instructional policies have argued that assessment and accountability have de-professionalised mathematics teaching and discouraged effective teaching (Darling-Hammond, 1997).

A deeper reflection on my assignment indicates that teachers' professional autonomy was and is still curtailed by the influence of external examinations. My autonomy as a mathematics teacher includes the freedom to decide how I teach and also what I teach (Helsby and McCulloch 1996).

In the course of offering this critique I came to the conclusion that whilst the aim of the National Curriculum was to embrace the 'professional' opinion of mathematics teachers, it was still fundamentally influenced by policy makers and examination boards that places pupil statistical achievement data above all other sources of evidence. This assignment posits the view that a greater understanding about the effectiveness of a school might be seen, if the inspection process placed higher value upon the professional integrity of mathematics teachers. The writing of this assignment provided me with a valuable opportunity to reflect upon how my professional practice had been fundamentally shaped by a political agenda that places greater emphasis upon test results than upon the collaborative professionalism of mathematics teachers.

Methods of Enquiry 1

The title of my assignment was, “An investigation on how I can use Realistic Mathematics Education (RME) strategy to improve my teaching of functional skills”. The assignment focused on how to design a research proposal. I used the assignment to gain a deeper understanding of issues that pertain to the application of mathematics across the school curriculum. How much link was there between the mathematics subject, other subjects on the school curriculum and real life experiences? These are the questions I carried with me as I went through the EdD programme. How can students learn mathematics in order to be functional in the community? I gained an insight into how to design research questions and how to establish theoretical frameworks relevant to the area of enquiry.

Specialist Course:

This assignment analysed the development and organisation of mathematics in 14–19 education over the last two decades. Assignment title: “*An analysis of the provision of mathematics in Post Compulsory Education and Training and its role in the New Diplomas*”. As a starting point, I looked at varying views of what mathematics is and what society expects of GCSE graduates. The knowledge and insight gained in this assignment became very valuable when I joined Higher Education in 2012. I began to fully appreciate the provision of mathematics in Post Compulsory Education and Training. Through this assignment, I sharpened my skills of reviewing and critically analysing literature.

Methods of Enquiry 2

The aim of this research was to examine and describe the role of mathematics in the teaching of Design and Technology in secondary schools in England. The research helped me to recognise a gap between current approaches to teaching mathematics and its application, not only outside the classroom, but within other subjects on the school curriculum. The principles that underlined this research were strongly influenced, by constructivists concerns that teaching of mathematics should occur within a context that is more meaningful to students (Clements 1987). The main focus was on whether students’ lack of mathematical skills acted as a barrier in the teaching of Design and

Technology? Whether teachers find students ready to apply their mathematical skills in Design and Technology? Focus was also on teachers' pedagogical views on whether mathematics teaching in the school is consistent with the requirements in their subject area? Participating teachers acknowledged the impact of students' mathematics knowledge on the teaching of design and technology. Different teachers indicated that they used different approaches to deal with mathematics so that it does not act as a barrier to students' progress in their subject.

The Institute Focused Study (IFS):

The IFS looked at the school's technological landscape focusing on electronic resources which were available in the school; how frequently teachers used them; teachers' perceptions of the usefulness of each resource and to what extent teachers' pedagogical practices had changed as a result of this landscape. Two theoretical frameworks (Levels of technology Innovation and Instrumental approach) were used to aide this investigation. Study sample comprised of 4 female and 2 male teachers from the mathematics department at the College. Study results showed that despite the availability of a wide range of electronic resources at the college, teachers used more instructional software resources than tool software that could promote high-order thinking skills. However, methods of integrating the same resource differed from teacher to teacher. The study also found that, though teachers expressed confidence in using different types of electronic resources, this did not translate to how frequently they used these resources. There was no evidence, however, to show the extent to which electronic resources have influenced teachers' pedagogical styles. The study helped me to identify a gap in the use of technology in schools and colleges, which I exploited in my thesis. This gap was focusing on students' experiences in using technology.

The thesis examines the experience of learners using GeoGebra to investigate modulus functions within a constructivist paradigm. It presents a reflective journey into the contributions of technology in understanding modulus functions and concentrates of learners' experiences while using GeoGebra. The contributions of technology in the teaching and learning of mathematics were examined firstly by exploring literature that deals with the use of dynamic mathematics software in teaching and learning

mathematics. My work on the thesis reflects my journey through all the modules completed during the EdD programme. These are manifest in the selection of theoretical frameworks, qualitative methods of data collection and analysis of data. The study finds that GeoGebra-assisted instruction, as a supplement to traditional classroom instruction, allows participants to explore aspects of modulus functions beyond the confines of the A Level Core 3 specifications.

Reflecting on my progression through the EdD programme indicates that the assignments acted as building blocks towards the construction of the final thesis report. They all focused on mathematics teaching and the role of technology in the teaching and learning of mathematics. Each time I worked on an assignment, designing research questions, I gained more information which influenced my initial perceptions and knowledge about pedagogical practices in mathematics. Most difficult in each assignment was to come up with a topic and research questions that would link with the previous work, therefore this demanded wider reading. Assignments helped me to understand the status of technology in mathematics education. I gained knowledge on how to search for literature, write research problems, identify theoretical frameworks, how to analyse the literature and form a coherent argument.

Receiving assignment feedback was one of the most important aspects of my studies. Through the feedback process, I saw myself as others see me. Feedback was a reaction by tutors, usually in terms of their knowledge and perceptions, telling me how my presentations could improve.

The journey through this research study has reshaped my vision on the use of technology in the classroom. The EdD programme made me realise that technology integration in mathematics education is not much about how many websites are accessible to students, or how organised a PowerPoint presentation is or about my own knowledge of the technology. It is the application of technology in ways that help students in the Zone of proximal development (see Vygotsky 1978) to achieve more than they could without technology. Working within a GeoGebra showed me how much knowledge students gained within a short space of time and how much they achieved, which would not have been possible without the use of GeoGebra. I now realise the

power of technology in teaching and I use digital technologies in mathematics, which allows students to explore and discover their own knowledge.

List of module assignments:

1. **Foundations of Professionalism:** *The national curriculum: a constraint or positive prompt to mathematics teacher professionalism.*
2. **Methods of Enquiry 1:** *An investigation on how I can use Realistic Mathematics Education (RME) strategy to improve my teaching of functional skills.*
3. **Post Compulsory Education and Training (PCET):** *An analysis of the provision of mathematics in Post Compulsory Education and Training and its role in the New Diplomas*
4. **Methods of Enquiry 2:** *The role of mathematics in the teaching and learning of Design and Technology.*
5. **Institute Focused Study (IFS):** *Integrating electronic resources in mathematics teaching: A case study of a secondary school in England.*
6. **Thesis:** *The contributions of Interactive Dynamic Mathematics software in probing understanding of mathematical concepts: Case study of a GeoGebra learning environment for the concept of modulus functions with A - Level students in England.*

The contributions of Interactive Dynamic Mathematics software in probing understanding of mathematical concepts: Case study on the use GeoGebra in learning the concept of modulus functions.

CHAPTER 1

In this Chapter, I outline the broad theme and purpose of the study alongside a description of the context that led me to develop an interest in this field. The chapter includes the broad aim of the research, which informed the basis of the review of literature contained in Chapter 2. I briefly highlight research studies on the use of technology in mathematics teaching and learning and the theoretical perspectives selected for the study. A detailed review of literature on dynamic mathematics software, GeoGebra and the modulus functions is discussed in Chapter 2.

1.0 INTRODUCTION:

This study explored how dynamic mathematics software package called GeoGebra, contributed to participants' learning and understanding of mathematics. The research focused on participants' experiences as they used and transformed GeoGebra to support their understanding of modulus functions. It highlighted participants' perspective on the role of GeoGebra in supporting exploration of modulus functions in particular and mathematical ideas in general. The focus of the study was not much, on how using GeoGebra enabled effective or efficient learning of mathematics, but on how participants utilised GeoGebra to address misconceptions and perceptions of modulus functions. The interactive capabilities of GeoGebra challenged me to create learning activities that engaged participants and encouraged more exploratory approaches to learning, where the participants were initiators and controllers of the construction of knowledge on modulus functions. The main research questions that guided this study focused on how students used GeoGebra to support their understanding of modulus functions and how GeoGebra related and contributed to their whole learning experiences.

The study found that GeoGebra provided a medium for visualisation that linked the abstract aspects of modulus functions with graphical illustrations. The study also noted

that working with GeoGebra extended participants' understanding of modulus functions i.e. (a) in distinguishing the difference between the graphs of $y = |f(x)|$ and $y = f(|x|)$ and being able to sketch the two graphs without the aid of GeoGebra; (b) challenging why the integral of $1/x$ is $y = \ln |x| + c$ instead of $y = \ln(x) + c$ since the derivative of $f(x) = \ln(x)$ is $1/x$; (c) recognising that not all turning points are stationary; (d) recognising the importance of graphical illustrations when solving equations and inequalities that involve modulus functions, to eliminate spurious answers derived from some algebraic calculations.

1.1 BACKGROUND TO STUDY:

The motivation for this study came from my own experiences as a mathematics teacher in schools and a lecturer in Further Education in England. The selection of modulus functions, as a topic, was instigated by: (i) its presence in the 'A' Level mathematics curriculum; (ii) its investigatory nature when solving equations and inequalities; (iii) the predominant role of graphical representations; (iv) the need to relieve pupils of work intensive technical operations involving graph sketching and (v) misconceptions identified in research literature (Monaghan and Ozmantar 2006) plus my own experience of teaching modulus functions at this level.

Ferrara et al (2006) notes that functions in general and modulus functions in particular are difficult for students to understand and conceptualise. In my own experience of teaching mathematics at 'A' level, I observed that students struggled to answer questions related to modulus functions. Students rarely used graphical illustrations when solving equations or inequalities that involved modulus functions. Monaghan and Ozmantar (2006) observed that students could not distinguish the difference between the graphical representations for the functions $y = |f(x)|$ and $y = f(|x|)$. They noted that students had the perception that all graphs of modulus functions looked the same when constructed from the graph of $y = f(x)$.

My second motivation was the need to integrate technology in my teaching. My Institute Focused Study (IFS: 2010), in partial fulfilment of my doctorate studies, focused on the extent of technology integration in mathematics teaching at a secondary school in England. I found very little evidence of technology integration in the teaching of

mathematics. Teachers mainly used PowerPoint presentations and mathematics websites for setting pupils' homework on Moodle. Moodle is a free open-source learning platform for producing modular internet based courses. Teachers used interactive whiteboards mainly as a replacement of the chalkboard and the overhead projector.

In a bid to improve the teaching and learning of mathematics, the school (studied in the IFS) had invested heavily in computer software and hardware specific to mathematics teaching and learning. However, this was not backed up with staff development courses on how to use the technology. I observed that there was very little use of the dynamic mathematics software resources like Geometer Sketchpad and Autograph, for which the school paid annual subscription fees. My observations supported Ofsted (2012) report, under the heading 'Under-exploited ICT and Practical Resources', which noted that technology use within school mathematics is still teacher-led. Teachers mainly used presentational tools such as PowerPoint, revision, practice and homework websites and very little of the dynamic mathematics software resources. Hedberg (2007) also noted that teachers perceived technology primarily as presentational and visual rather than as an instrument to facilitate mathematical thinking and reasoning.

I chose the dynamic mathematics software, GeoGebra, due to its accessibility during normal mathematics lessons and outside lesson time on students' iPhones, laptops and personal computers. In addition, there were many GeoGebra resources, related to modulus functions, readily available to teachers and pupils on the college portal. The college worked closely with MEI (Mathematics Education for Industry). MEI is a UK-based mathematics education charity, which supports teachers through a range of professional development opportunities and creates innovative teaching and learning resources. MEI became an Institute of GeoGebra in 2013. This move was intended to provide schools with free access to high quality training and resources for GeoGebra. Due to the college's close association with MEI, there were lots of GeoGebra resources available to students in a variety of topics. However, at the time of undertaking this study, very few of the GeoGebra resources were being utilised by students and lecturers.

The third motivation came from a review of government initiatives, national policies and new 'A' level mathematics curriculum for England and Wales (2013). Government initiatives on reforms in mathematics education suggested major changes in the teaching of mathematics. It recommended learners to become actively involved in constructing their own knowledge and developing mathematical concepts as they explored, explained, and justified solution strategies to mathematical tasks. For more than a decade, the Department for Education and Skills (DfES), which existed until 2007 and Ofsted (2008 and 2012), advocated for the use of technology in the teaching of mathematics to allow pupils to explore and learn mathematics through experiments and observations. Ofsted report (2012): *'Mathematics Made to Measure'*, drew attention to the underdeveloped use of ICT, at school level, to enhance learning and promote pupils' problem solving skills. The educational reforms in 'A' Level mathematics focus on using technology to develop problem solving skills among students. Government initiatives and national policies emphasise students' development of high-order thinking and problems solving skills. I realised a need to take the challenge of using technology to learners. This study focused on learners' experiences within a mathematics classroom.

The fourth and last motivation for this study was to address the gap in research literature on students' interaction with technology to develop an understanding of mathematical concepts. This study adds to the growing list of research literature that focuses on the use of technology in the teaching and learning of mathematics. There is research on the use of technology in teaching mathematics, but most has focused on undergraduate mathematics concepts such as calculus (Serhan 2009, Nardi 2008); using dynamic geometry software at secondary school level (Aymemi, 2009; Mehanovic, 2011); and GeoGebra in the teaching of primary school mathematics (Chrysanthou, 2008). This leaves a gap on fundamental concepts of modulus functions at 'A' Level. The study focused on learners' experiences of using GeoGebra to investigate the concept of modulus functions. It explored how GeoGebra contributed to their learning and understanding of modulus functions at 'A' level.

1.2 PURPOSE OF THE STUDY:

The main purpose of this study was to explore how dynamic mathematics software package called GeoGebra, contributed to participants' learning and understanding of mathematics.

In spite of the great expectations expressed by many educators and researchers in the last two decades, it is hard to ascertain the extent to which technology has been really integrated in mathematics learning at school level. Most of the literature available (discussed in detail in Chapter 2) offers a wide range of theories, methodologies and interpretations, which are often related to the potentialities of technology for mathematics education, rather than pupils' experiences in using technology in their learning. Research is dominated by publications about innovative use of different types of software and their applications in teaching and learning mathematics (Lagrange et al 2003).

In England, as far back as 1999, the Teacher Training Agency (TTA) cited in Chrysanthou (2008:12) offered a rationale for making use of ICT to support children's learning of mathematics. TTA suggested then, that information and communication technology (ICT) had the potential to make a significant contribution to pupils learning mathematics. ICT could help learners to: explore, interpret and explain patterns...; develop logical thinking; and develop mental imagery. Van Voorst (1999:2) highlights that "technology helps students to visualise certain mathematical concepts better..." and that "it adds a new dimension to teaching of mathematics".

Literature, as will be discussed in Chapter 2 is very positive and rarely negative about the impact of technology use in the teaching and learning of mathematics. Research on dynamic mathematics software focuses on technological and pedagogical issues, largely overlooking learners' experiences in using the technology. Sharpe et al (2005) argue that many studies have focused primarily on pedagogy and teachers' experiences rather than on how learners actually use and experience technology.

Other research studies have been conducted in order to find out more about technology integration from an instrumental genesis perspective (Rabardel 1995, Artigue, 2002,

Ruthven 2002 and Trouche 2005). Bartolini Bussi (2001), Mariotti (2002), Duval (2006), Hoffmann (2001) and Mariotti and Bartolini Bussi (2008) focused their research studies on different types of educational mathematics software like CAS and dynamic geometry software from a semiotic mediation theoretical perspective. They highlighted the potential of dynamic mathematics software in improving the teaching and learning of mathematics.

However, to date there is limited research evidence that acknowledges learner perspective in the development of tools, pedagogy and learning experiences while using technology. GeoGebra, the focus of this study is a recent addition to the new technological tools to find its way into the classrooms. Being generally available just recently, there are comparatively a good number of studies (Hohenwarter 2004, Hohenwarter et al 2006, 2007, 2008, Dikovic 2009, Bayazit & Aksoy 2007, Mehanovic 2011, Mainali & Key 2012) that have looked at GeoGebra and its use by learners in classroom settings. This study adds to the growing research work on technology use in learning but with a special focus on 'A' Level participants and their learning experiences as they work with GeoGebra.

I carried out this study in the role of a practitioner-researcher with a group of six 'A' level students in their second year of study. Whilst this cannot be deemed representative, nevertheless, the study can claim to respond to the need for insider-researcher's perspective on how GeoGebra can be utilised in an ordinary mathematics classroom. The topic chosen for this study was modulus functions and it is chosen from the Core 3 'A' Level mathematics curriculum for England and Wales. More specifically, this study observed and investigated participants' experiences of using GeoGebra to understand the concept of modulus functions.

1.3 RESEARCH QUESTIONS:

The primary research questions that guided this research are:

1. In what ways did participants use GeoGebra to support their understanding of the concept of modulus functions?
2. How does GeoGebra relate to and contribute to the whole learning experience?

In order to explore in depth ways in which participants used GeoGebra to support their understanding of the concept of modulus functions, a secondary research question was created for research question 1: Does the use of GeoGebra increase speed and efficiency of the process of sketching graphs involving modulus functions and improve the accuracy and presentation of results in participants' written work? I used Ruthven and Hennessey's (2002) model to highlight the contributions of digital tools and resources in mathematics teaching and learning.

Research question 2 was sub-divide into three sub-research questions:

- a. Is there any evidence of participants using GeoGebra to support processes of checking, refining and improving strategies and solutions?
- b. Does the use of GeoGebra overcome participants' difficulties and enhance their sense of capability and confidence to tackle difficult tasks?
- c. Does the employment of GeoGebra foster participant independence and peer exchange, notably by providing opportunities for pupils to share knowledge and provide mutual support?

Research questions 1 and 2 split the study into four specific aspects. The first aspect was to establish a participant-perspective of GeoGebra as a software and tool in the learning environment. The second aspect was to investigate how participants used GeoGebra to support their learning. The third aspect was to identify approaches or strategies used by participants while learning with GeoGebra. The fourth and final aspect was to identify how participants integrated GeoGebra in their normal learning activities. These four aspects guided me in the selection of theoretical frameworks, data collection and data analysis methods.

I selected the instrumental genesis and semiotic mediation theoretical frameworks. Instrumental genesis framework provided the theoretical lens through which I made sense of how learners used GeoGebra and how GeoGebra contributed to their understanding of mathematics. Semiotic mediation provided the means to understand how participants used the graphical and symbolic algebraic representations to construct knowledge of modulus function. These frameworks are discussed in detail in Chapter 2.

The two frameworks were supplemented by Goos et al's (2004) four metaphors, which theorise the varying degrees to which students use technology. Goos et al's (2004) carried a three year longitudinal study that investigated the role of electronic technologies in students' exploration of mathematical ideas and in mediating their social interactions with teachers and peers. Their study explicitly addressed technology as a tool that is integral to the mathematical practice of students and teachers in particular learning environments. Goos et al (2004) came up with four roles for technology in relation to teaching and learning interactions (master, servant, partner and extension of self). These metaphors are discussed in detail in Chapter 2: Section 2.1.3.

I examined in detail the experience of the participants while using GeoGebra, as they worked through set tasks during mathematics lessons. This study worked with a small group of participants with the aim to interpret the experiences of each individual while working in a GeoGebra environment. I carried out observations during scheduled workshop sessions. The workshops are slots on the main timetable aimed at giving students additional support in their learning.

Participants' activities and discussions were captured on screencast videos and analysed to aggregate frequently used features of GeoGebra; the ways in which participants used the software; the discussions; conjectures derived from their observations of multiple representations and the approaches they used to solve problems.

In the last session, I elicited for participants' views about their experiences of using GeoGebra during the sessions. Open questions were raised during these discussions. The purpose of open questions was to provide an opportunity for participants to describe their learning experiences and how they used GeoGebra. Lesson observation notes, data from screencast video recordings and group interview was triangulated based on themes derived from the research questions.

Since the research focused on participants' experiences of investigating a mathematical concept using GeoGebra, data collected through screencast video clips and field notes was analysed using an interpretive phenomenological analysis approach. Interpretive

phenomenological analysis (IPA) is a recently developed approach to qualitative enquiry committed to examining how people make sense of their major life experiences (Smith et al 2012). I selected it for this study because it is concerned with exploring experience of participants in its own terms. Smith et al (2012) define phenomenology as a philosophical approach to the study of experience. The founding principal of phenomenological inquiry, which has its roots in the field of healthcare research, is that experience should be examined in the way that it occurs and in its own terms.

This study is interpretive in the sense that participants recorded their activities and gave an account of their experiences during interviews. As a researcher, I interpreted participants' interpretations of their own experiences. I also analysed and interpreted activities recorded on screencast video clips as well as responses from interview data, in order to understand their experiences. I tried to make sense of participants, making sense of what was happening to their learning process. Interpretive phenomenological analysis is committed to a detailed examination of a particular case (Smith et al 2012).

1.4 OVERVIEW OF THE STUDY:

This research is rooted in the theoretical framework of a Vygotskian perspective focusing on the construction of knowledge and on the mediation of tools accomplished through technology. The research therefore adopted a case study approach within a qualitative design to investigate how participants used dynamic mathematics software to develop an understanding of mathematical concepts. This approach allowed me to get in-depth understanding of the experiences of participants as they used dynamic mathematics software to explore the concept of modulus functions. The selected software for this study was GeoGebra. I focused on a small group of participants, working within a GeoGebra environment to learn about the concept of modulus functions.

The research also sought to expand the discourse of how students employ technology in their mathematical studies or more explicitly how they adapted and moulded GeoGebra for their own needs.

The literature review in Chapter 2 focuses on literature particularly relevant to this study. I divided Chapter 2 into four sections. The first section discusses the theoretical framework chosen for the study. The second section reviews literature on mathematics learning while using dynamic mathematics software. The third section analyses and discusses the functioning of specific elements of GeoGebra as semiotic registers in classroom activities. The fourth and final section looks at the concept of modulus functions as stipulated in the 'A' Level curriculum.

Chapter 3 describes the methodology applied to answer the following research questions: (1) How students employed GeoGebra to support their understanding of modulus functions and their properties? (2) How GeoGebra related to and contributed to the whole learning experience? It discusses in detail the research methodology used, research design selected, participants' profiles and how the sample was selected. It also covers methods of data collection, data analysis methods and ethical considerations.

Chapter 4 outlines, describes and comments on the activities carried out during six sessions of the study. In each session, I highlighted how participants used technology (in this case, GeoGebra) to learn different aspects of modulus functions particular to each session. Session 1 was divided into three activities. It focused on the introduction of GeoGebra, while at the same time working with problems that involved modulus functions. Session 2, dealt with the investigation of graphs of polynomials involving modulus functions. In session 3 participants sketched and compared the graphs of $y = |f(x)|$ and $y = f(|x|)$. Session 4 looked at equations involving modulus functions. Session 5 was a follow up to session 4 but focused on solving inequalities involving modulus functions. In the final session, we carried out a group interview in the form of a discussion. An exercise, based on past examination questions, was also given to check participants' understanding of the concept of modulus function. Their answers and presentation of work was analysed.

Data consists of extracts from screencast video clips captured during sessions, field notes carried out during lesson observations and class discussions, pencil and paper sketches made by pupils in some activities and interviews carried out with participants

during and at the end of the study. Data is analysed using an interpretive phenomenological analysis framework (Smith et al 2012).

Chapter 5 is a summary of research conclusions and discussions. The first section contains answers to the research questions and summarises those results that are relevant to the teaching of modulus functions in a GeoGebra environment. The second section discusses the research findings and outlines some highlights of the research and recommendations.

Chapter 6 outlines the contributions and implications of the study. It also discusses the limitations of the study.

CHAPTER 2:

2. REVIEW OF LITERATURE:

Chapter 1 briefly outlined the background to the study, research questions, and purpose of the study, with the ultimate aim to investigate how dynamic mathematics software supports the learning of mathematical concepts. This chapter outlines the theoretical frameworks used and how they contributed to the study. It outlines the role played by technology and multiple representations in the process of learning. The review looks at the contribution of dynamic mathematics software packages in developing and understanding mathematical concepts in general. It focuses on various studies that have investigated the use of technology in the teaching and learning of mathematics in general and algebraic functions in particular. The review served to highlight gaps in the literature where this study could fit.

2.0 Literature review:

Marshall and Rossman (1999) cited in Creswell (2008) and Cooper (1984) argue that reviewing literature relates the study to a larger and on-going dialogue about a topic, filling in gaps and extending prior studies. The study referenced different pieces of research, which in my opinion constituted a coherent set focusing on issues relevant to this study.

Review of literature related to this study provided a framework for establishing the layout of the study and benchmarks for comparing results with findings from other studies that investigated the use of dynamic mathematics software in the teaching and learning of mathematics. It also gave me an insight on how other comparable studies selected methodologies, methods and theoretical frameworks and how data was analysed (Gray 2009). This informed and guided my choice of methods of data collection, presentation and analysis.

The literature review, therefore, concentrates on five main areas of research relevant to this study. The first section describes the framework used to conduct the research and to analyse the data for this study. The second section examines previous studies that focused on integration of dynamic mathematics software in mathematics teaching and

learning and highlights their findings. The third section focuses on the role of multiple representations in the teaching of mathematics in general and algebraic functions in particular. The fourth section focuses on the software, GeoGebra and includes a brief history of its development, its features and why it was selected for this study. The fifth and last section focuses on modulus functions: their place in the A level curriculum in England and Wales, their nature and research studies that have investigated the teaching of modulus functions.

2.1 THEORETICAL FRAMEWORK:

The field of technology use in the teaching and learning of mathematics has over the years, presented a theoretical challenge to researchers. Various conceptual frameworks that potentially provide an insight into technology in the learning and teaching of mathematics have been proposed. Research studies on technology integration in mathematics education (Verillon & Rabardel 1995; Rabardel 1995; Guin & Trouche 1999; Lagrange 1999; Artigue 1998; Artigue 2002; Trouche, 2005), have taken place, primarily from a constructivist perspective on learning, which emphasises the social and cultural nature of mathematics activities. The constructivist approach to learning argues that learning is personally constructed and is achieved through interaction with tools / artefacts (Kafai & Resnick 1996).

Constructivist learning theory is founded in the research of Jean Piaget. Piaget's (1973) central concern was not whether children construct knowledge but rather the process through which they construct knowledge. Recent views of constructivist learning in mathematics education research add to the work of Vygotsky (1978). Vygotsky believed that individuals could achieve higher ground through interaction with more knowledgeable individuals.

Vygotsky pointed out that in the practical sphere, human beings use artefacts, reaching achievements that would otherwise have remained out of reach. My perspective of constructivist learning approach combines views of Piaget and Vygotsky. My perspective evolved as a result of reviewing literature and reflecting on my own teaching practices.

Individuals construct knowledge and social interaction facilitates that knowledge construction. This research, therefore, is rooted more in a Vygotskian socio-constructivist perspective focusing on how learners construct knowledge and how GeoGebra, as an artefact in a learning environment, mediates in the learning process.

2.1.1 Philosophical Paradigms: Objectivism and Constructivism:

Objectivism has dominated the field of education since the time of B.F. Skinner and Robert Gagné. Vrasidas (2000) argue that most of the traditional approaches to learning and teaching based on behavioristic and cognitive theories share philosophical assumptions that are fundamental in objectivism. According to Vrasidas (2000), the objectivist believes that there is a real world out there consisting of entities structured according to their properties and relations. They believe that real world is fully structured and that it can be modeled. Knowledge and truth exist outside the mind of the individual and are therefore objective (Tam 2000; Jonassen 1991a). This implies that knowing and learning are processes for representing and mirroring reality. Individuals can interpret either knowledge as correct or incorrect (Jonassen 1991).

Objectivists regard thinking as effective only if it adequately describes some objective reality. The position that objectivism makes is that the world is real, that it is structured, and that its structure can be modeled for the learner. The major assumptions of objectivism are that: knowledge exists outside the learner and there is an absolute truth; learners are told about the world and are expected to replicate its content and structure in their thinking. It asserts that there is a particular body of knowledge that needs to be transmitted to a learner. Learners are not encouraged to make their own interpretations of what they perceive the reality to be (Jonassen 1991).

The role of education in the objectivist view is to help students learn about the world. Learning is, thus viewed as the acquisition and accumulation of a finite set of skills and facts (Tam 2000).

The philosophical and epistemological assumptions of objectivism are then contrasted with constructivism. Constructivism is a philosophical and epistemological approach, which describes learning as a change in meaning constructed from experience (Newby

et al. 1996). It places learners at the centre of the learning process. Constructivism holds that knowing, is a process of actively interpreting and constructing individual knowledge representations. At the heart of constructivism, is a concern for lived experience or the world as experienced and understood by the learner. This is radically different from what objectivists conceive to be learning.

Constructivism is concerned with how learners construct knowledge. Simon (1995) argues that constructivism derives from a philosophical position that human beings have no access to an objective reality, that is, a reality independent of our way of knowing it. Human beings construct knowledge of the world from their perceptions and experiences, which are themselves, mediated through previous knowledge. Constructivism acknowledges the existence of the external reality. However, Vrasidas (2000) argues that this external reality only sets boundaries of what people can experience. Vrasidas (2000) concurs with Jonassen (1991) who argues that reality is local and people construct their own reality through interpreting perceptual experiences of the external world. No one true reality exists, except for individual interpretations of the world. Interpretations are shaped by experience and social interactions.

In this study, I used a constructivist epistemology as a philosophical approach to investigate the scope, structure and very nature of knowledge constructed through participants' interaction with GeoGebra.

2.1.2 The constructivist approach:

Constructivist theory has been prominent in recent research on mathematics learning and teaching and has provided a basis for recent reforms in mathematics teaching. In the 90s, constructivist approach to teaching was central to empirical and theoretical research in mathematics education carried out by Steffe and Gale (1995), Cobb (1995), von Glasersfeld (1991), and Cobb, Yackel & Wood (1992) to mention a few.

Constructivism focuses attention on how people learn. Recent efforts to integrate technology in the classroom have been within a constructivist framework (Gilakjani et al 2013). Jonassen (1991) argues that in a constructivist-learning environment, students use technology to manipulate data and explore relationships. It suggests that

mathematical knowledge results from learners forming models in response to tasks and challenges that come from actively engaging mathematics problems and environments – not from simply taking in information.

There are two constructivist perspectives that have dominated research on technology in mathematics education: (1) radical constructivism and (2) social constructivism. Cobb (1994) argues that the radical and social constructivism cannot be separated because both complement each other. The two perspectives share the same epistemological assumption that knowledge or meaning is not discovered, but constructed by the human mind. Knowledge is actively created or invented by the learner, not passively received from the environment (Cobb 1994). Students need to construct their own understanding of mathematical concepts.

The major difference between radical and social constructivism is around the locus of knowledge construction. Even though both perspectives have the same general interest on how individuals learn or construct knowledge, they differ markedly with respect to the mechanisms they see at work.

Radical constructivism is associated with Ernst von Glasersfeld, whose thinking was profoundly influenced by the theories of Piaget. von Glasersfeld (1989) defines radical constructivism according to the conceptions of knowledge. He sees knowledge as being constructed by the learner through the senses or by way of communication (von Glasersfeld 1989). von Glasersfeld assumes that external reality cannot be known and that the learner constructs all knowledge ranging from everyday observations to scientific knowledge. von Glasersfeld focuses on the individual learner and pays scant attention to the social processes in knowledge construction.

Radical constructivists argue that it is not possible to judge knowledge as an ontological or metaphysical reality (Terhart 2003). It emphasises discovery learning in complex situations and learning in social contexts, while strongly distrusting systematic evaluation of educational outcomes. Radical constructivists argue that students must discover knowledge by themselves without explicit instruction. It focuses on the individual's construction of knowledge, thus taking a cognitive or psychological

perspective. Simon (1995) argues that even though social interaction is seen as an important context in radical constructivism, the focus is on the reorganisation of individual cognition.

This research is rooted in the social constructivist theoretical perspective, with particular regards to the social construction of knowledge through the mediation of technology. The selection of a constructivist epistemology was influenced by social constructivist perspective, which assumes that learning precedes intellectual development through mediated transactions and that social interaction plays a fundamental role in the children's acquisition of knowledge (Cole 1996, Wertsch 1991).

Social constructivism views learning as a collective process by which people learn through interaction with artefacts, signs, other peers and adults (Goos et al 2001). It focuses on how individuals construct and make sense of their world (Robson, 2009). Social constructivists argue that the formation of learning requires, among other things, the use of technology and symbolic tools (Artigue 2002) to act as mediators of knowledge acquisition.

Vygotsky, one of the proponents of social constructivism, focused on the social factors that influence learning. In Vygotsky's days, there was no digital technology, which has radically changed the learning environment from what he originally conceived it to be. The introduction of technology has seen a number of researchers (Ridgeway 1997, Verillon & Rabardel 1995) revisiting Vygotsky's socio-constructivist theories with the view to reconceptualise them within the emerging research field of technology in mathematics education (Clark-Wilson 2010). Developments in technology have brought in new dimensions to warrant technology inclusion within the domain of cultural artefacts available in mathematics learning environments. Crawford (1996) considers technology as cultural artefacts that students can use to mediate and internalise mathematics learning.

The social constructivist perspective adopted for this study entails understanding how interaction with GeoGebra shaped the development of the learners' conception of modulus functions in General Certificate of Education (GCE) Advanced level

mathematics, how they interacted with the software and what they learned as they worked in this environment.

Kaptelinin (2005) claims that the nature of any artefact can be understood within the framework of human activity, by understanding the ways people use this artefact and the needs it serves. Kaptelinin (2005: 56) argues that, "... technology is just another artefact that mediates the interaction between learners and their learning environment". The human activity perspective states that computer technology does not only change the tasks but also often empowers the individual learner to develop cognitive structures that remain in the mind even if the external tool is no longer used (Kaptelinin 2005).

The research follows two major theoretical frameworks within a socio-constructivist perspective: (a) instrumental genesis which draws on Vygotsky's (1978) work on tool use and; (b) semiotic mediation perspective (Bartolini Bussi & Mariotti 2008, Nardi 1996), which looks at the potentiality of artefacts in fostering a relationship between pupils and mathematical knowledge. The next section explains how the two perspectives (instrumental genesis and semiotic mediation) provide a unified theoretical framework in examining the learning process in a GeoGebra integrated mathematics classroom.

2.1.3 Instrumental approach:

Instrumental genesis, attributed to social constructivist Rabardel (1995), draws on Vygotsky's (1978) work on tool use. The complex nature of tool use has acquired considerable attention within cognitive ergonomics (Rabardel 1995) as well as the activity theory Engeström (1999). The instrumental approach focuses on the way individual learners use tools (Artigue, 2002; Rabardel, 2002; Trouche, 2005; Verillon & Rabardel, 1995). The approach stresses the distinction between the artefact – the object and the instrument.

For the purpose of this study, words 'artefact' and 'instrument' are often used interchangeably to mean the same thing. However, the distinction between artefact and instrument helped me to analyse separately the potentialities of GeoGebra and the actual uses that were realised through it, as explained below.

Rabardel (1995) defines an artefact as the material symbolic object designed according to a particular goal, embedding a specific knowledge. He defines an instrument as a mixed entity made up of both artefact-type components and schematic components. Instruments are a construction of an individual with a psychological character strictly related to how an artefact is being utilised within a context. Vygotsky (1981) cited in Clark-Wilson (2010) defines artefacts as artificial devices for mastering mental processes and offers a list of examples, which include language, counting systems, algebraic symbols and diagrams.

Rabardel (2002) views an artefact as a particular object with intrinsic characteristics, designed and realised for the purpose of accomplishing a particular task. Artefacts are artificial objects that have been invented for certain purposes. To use a specific artefact, the learner needs to develop schemes, progressively. In the analysis of educational software used in this study, I initially treated the elements of the interface of GeoGebra as artefacts and observed them being developed into instruments as participants worked through modulus function activities.

Rabardel (1995) argues that an instrument is more than an artefact. It consists of the artefact and cognitive schemes, needed to use the artefact. An instrument is the association of an artefact and the modalities of its use, as elaborated by the user (Rabardel 1995). Rabardel further argues that learners transform an artefact into an instrument in order to perform a task. In the case of this study, an instrument is defined as the artefact used to perform the task plus the learners' schemes, which allow them to perform the task and control the learning activity.

Rabardel (1995) refers to the process of building an instrument from an artefact as instrumental genesis. Guin and Trouche (1999) describe instrumental genesis as the process by which an artefact is transformed into an instrument by the user. An artefact at the outset does not have an instrumental value. It is a material or abstract object, given to a user. An artefact becomes an instrument through a process, or a genesis, by the construction of cognitive schemes (Verillon and Rabardel 1995).

Instrumental genesis consists of two interrelated processes: the instrumentation process directed towards the user and the instrumentalisation process directed towards the artefact (Verillon and Rabardel 1995).

Instrumentation process is directed towards the user, loading the artefact with potentialities and eventually transforming it for specific uses. An instrument is thus, made up of an artefact component and a cognitive component (knowledge necessary for using the artefact for performing the task). The process of constructing cognitive schemes is developmental since they emerge as users execute tasks (Verngaud 1996). Cognitive schemes are psychological constructs that operationalise an artefact in order to carry out specific tasks (Artigue 2002).

In the past, some psychologists (Verillon & Rabardel, 1995) have shown through empirical research, how by drawing on learners' previous knowledge and experience, artefacts give rise to mental construction of knowledge. While using an artefact in a mathematical activity, the artefact shapes the learner's mental conception of the activity and the learner builds his/her own schemes of action for the artefact (Guin & Trouche, 1999; Hoyles, Noss & Kent, 2004; Mariotti, 2002; Trouche, 2005). Trouche (2005) says that instrumentation comprises not only the rules and heuristics for applying an artefact to task, but also an understanding of the task through which that application becomes meaningful to the user.

Artigue (2002) defines instrumentalisation as the process when users discover the various functionalities of the artefact and transform them into personal use. It is concerned with the emergence and development of utilisation schemes. Instrumentalisation process depends on the user and leads to an internalisation of the uses of the artefact. Instrumentalisation can go through different phases: a stage of discovery and selection of the relevant functions, a stage of personalisation and a stage of transformation of the artefact sometimes in a direction unplanned by the designer.

Instrumental genesis therefore, comprises the development of cognitive schemes containing conceptual understanding and techniques for using an artefact for a specific task. The resulting instrument integrates the artefact and mental schemes. The user's

knowledge guides the way the tool is used and in a sense shapes the artefact into an instrument.

Researchers (Verillon & Rabardel, 1995; Rabardel, 2002; Guin & Trouche, 1999; Lagrange, 1999a; Artigue et al., 1998; Artigue, 2002 and Trouche, 2005) have used instrumental genesis framework to elaborate a mutual transformation of learner and artefacts in the course of construction of knowledge with technological tools. White (2008) analysed three classroom episodes, investigating the relationship between a designer's objectives and students' use of the tool in situated activities in a learning environment. White (2008:9) concluded that, "Instrumental genesis makes an artefact meaningful in the context of mathematical activities and provides a means by which users simultaneously make sense of the artefact and the concept being studied".

The instrumental genesis framework focuses on the mediating role of tools by stressing the co-emergence of tool techniques and meaning in the process of learning. In this study, the instrumental genesis framework addressed the first research question: "In what ways did participants use GeoGebra to support their understanding of the concept of modulus functions". I used the instrumental genesis framework to make sense of students' mathematics activities where GeoGebra was distinctively instrumental in the learning process. It provided an appropriate theoretical framework for analysing the process through which GeoGebra became a conceptual tool, simultaneously characterising the ways participants came to implement and understand the software in the context of learning the concept of modulus functions. Instrumental genesis dealt with the technology aspect of this study, focusing on how participants used GeoGebra.

To analyse instrumental genesis within a group context, I used Goos et al's (2004) four metaphors, which theorise the varying degrees to which students use technology. The instrumental genesis framework focuses on the way individual learners use tools (Artigue 2002, Rabardel 2002, Trouche 2005, Verillon & Rabardel 1995). Goos et al. (2004) focus on group interactions and whole class discussions where students use technology to share and test their mathematical understanding.

The four metaphors were drawn from a three-year longitudinal study of senior secondary school classrooms. The longitudinal study examined pedagogical issues in using technology as a mediator in mathematics teaching. From that study, Goos et al (2004) illustrated four roles for technology in relation to teaching and learning interactions. Firstly, they illustrated technology as a master. At the first level, students' knowledge and usage of the technology was limited to a narrow range of operations. Secondly, they illustrated technology as a servant. Students used technology as a supplementary tool but not in creative ways that changed the nature of activities. Thirdly, technology is illustrated as a partner. Students used technology creatively to provide new ways of approaching existing tasks, an observation also made by Templer, Klung & Gould (1998). Technology was also used to mediate collaboration and discussion during lessons. The last of the four metaphors was where technology was used as an extension of self. At this level, students used technology to extend their existing competencies. Technology became a natural part of their mathematical tools. Students became confident to use the technology to explore new ideas.

For meanings to emerge from the mathematical context, it was crucial to identify the relationship between the use of the artefact and the mathematical knowledge. Semiotic mediation framework discussed below, was the appropriate framework to analyse this relationship.

2.1.4 Semiotic mediation framework:

Semiotic mediation framework enabled me to understand the effects of multiple representations and their contribution to participants' learning experiences. Analyses such as those by Maracci and Mariotti (2009) cited in Drijvers, Kieran & Mariotti (2010) show that the semiotic mediation framework complements theories on instrumentation. Semiotic mediation deals with the potentiality of fostering a relationship between pupils and mathematical knowledge, through the means of representations.

Arzarello (2006) sees the combination instrument-artefact as a semiotic system in the wider sense of the term. An instrument is a semiotic representation of an artefact with rules of use that bear an intentional character (Arzarello 2006).

Over the last years, semiotic mediation has been broadly discussed as a theoretical perspective for analysing and describing some central problems of mathematics education. Falcade et al. (2007), Bartolini Bussi and Mariotti (2008) and Duval (2006) conducted research in the field of mathematics education for many years. They contributed to the development of a theoretical framework for semiotic mediation in the mathematics classroom with publications in several research studies.

In a research project that involved 10th grade classes (15 – 16 year old students) in France and Italy, Falcade et al (2007) investigated Cabri tools as instruments of semiotic mediation in the learning and teaching of mathematics. From excerpts of collective discussions during their project, they noted particular ways in which the ‘trace’ tool in Cabri functioned as a potential semiotic mediator but the report does not specify “How?”

In semiotic terms, learning mathematics is viewed as acquiring the competence to express oneself appropriately through the means of symbols or multiple representations (Hoffmann 2001). The process of semiotic mediation occurs when learners use an artefact to accomplish a mathematical task. Any artefact is a tool of semiotic mediation as long as it is intentionally used to mediate mathematical content. There are techniques related to the actual use of the artefact and techniques related to the mathematical tasks developed in the process. Mathematical knowledge gained by students through experience with technological tools is shaped, not only by the tasks they tackle or by their interactions with the teacher and other students, but also by the features of the artefact (Jones 2000). Cobb (1995) argues that the artefact that students use is central to the process by which students acquire knowledge and skills. They shape and transform mental processes. However, the function of semiotic mediation of an artefact is not automatically activated by merely using the artefact. For meanings to emerge, it is crucial to identify the relationship between the use of the artefact and the mathematical knowledge.

In a classroom setting, a mediated mathematical action is grounded in the way various tools such as mathematical symbols, computer generated diagrams, graphs, notation systems of software, iconic and graphic representations are appropriated. Rogoff (1990) claims that participating in mathematical activities, mediated by technology, enables the

students to internalise technology as a thinking tool and then move towards structural reorganisation of mind that makes it possible to think without such tools.

GeoGebra is a computational software with semiotic capabilities, because it provides a close connection between the symbolic manipulation and visualisation capabilities of Computer Algebra Systems (CAS) and Dynamic Geometry Software (DGS). The features of GeoGebra that can contribute to the learning of mathematics are seen much more clearly in the semiotic perspective than using the instrumental approach. GeoGebra has a very specific way of simultaneously using multiple windows to display the same concept. In its interface, all the windows (the CAS window, Graphics window, Spreadsheet, 3D graphics and algebra window) can be viewed simultaneously.

Duval (2006) defines semiotic registers as systems that permit transformation of representations from one format to another. A semiotic system for representations consists of a 'source register' (i.e. the Input window) and a 'target register' (i.e. graphics and algebraic screens in GeoGebra). According to Duval (2006), viewing multiple screens simultaneously creates an environment for the possible conversion of the same concept between the two semiotic registers. This helps learners unify concepts (Duval 2006). Various kinds of conversions between the registers/screens touch on the cognitive complexity of comprehension in mathematics learning and on the specific thinking processes required in a mathematical activity (Duval 2006). Comprehension assumes the coordination of at least two registers of semiotic representations (Duval 2006).

GeoGebra presented a semiotic system and each representation in the multiple windows, was considered, in this research, as a semiotic register in the sense defined by Duval (2006). The representations in the GeoGebra input window represents the source registers, with the graphs and algebraic expressions displayed in the multiple windows as the target registers. However, by using a mouse to drag the graph across the screen; selecting a tool from the menu; using commands from the 'Input Help' menu or using sliders to change value of parameters, can sometimes assume the role of source registers. In GeoGebra, the 2-D graphics register has a slider component that forms an environment for dynamic variations. Duval (2006) claims, variations in the

graphics register provided by the slider and the algebraic register are key elements in the construction of associated meanings in algebraic functions.

Duval argues that being able to convert representations between semiotic registers is the threshold of mathematical comprehension. Understanding and learning mathematics presupposes the ability to transform representations within given semiotic systems and to switch between the different registers Duval (2006).

The next section defines how the phrase ‘dynamic mathematics software’ is used in this study and highlights claims and findings from research literature on the dynamic mathematics software’s capabilities to enhance mathematics teaching and learning.

2.2 Dynamic mathematics software:

The term ‘dynamic mathematics software’ is used in this study to refer to Dynamic Geometry Software (DGS) such as Cabri and Geometer Sketchpad (which concentrates on relationships between points, lines and circles) and Computer Algebra Systems (CAS) such as Derive, Autograph and Maple, (which focus on manipulation of symbolic expressions). This study specifically focuses on GeoGebra, a hybrid software which combines functions from DGS and CAS. Hoyles and Noss (2003) view dynamic geometry systems as pedagogic tools finely tuned for the exploration of a mathematical domain and providing a setting in which students can construct and experiment with geometrical objects and relationships.

Dynamic mathematics software environments are intended to improve the learning of mathematics by creating a context in which sense can be made of a mathematical activity (Gutierrez, Laborde, Noss & Rakov 1999). Pioneering work by researchers in mathematics education has associated dynamic mathematics software with pedagogical orientation, in which it serves to create experimental environments where collaborative learning and student exploration are encouraged. Research in dynamic mathematics software: Computer Algebra Systems (Artigue, 2002, Ruthven, 2002), and Dynamic Geometry Software (Mariotti 2002, Laborde 2003), generally presents the software as a potentially important and effective tool in the teaching and learning of mathematics.

Ruthven et al. (2008) argue that dynamic mathematics software has the potential for changing the beliefs and the behaviour of learners. This in turn changes learners' view of mathematics from a static-deductive activity that emphasises proofs already discovered by others, to an exploratory-inductive activity that emphasises the heuristics involved in discovering results (Ruthven et al. 2008).

Mainali & Key (2012) claim that dynamic mathematics software offers new tools that go beyond pencil and paper methods. This concurs with Ruthven's argument that dynamic mathematics software enhances the discovery learning process by enabling students to explore more and more examples on the computer screen than is feasible with pencil and paper. The multiple related representations in dynamic mathematics software environments provide an opportunity for direct manipulation of both symbolic and graphic representations (Mainali & Key 2012).

Pitta-Pantazi and Christou (2009) only highlight the capabilities of dynamic geometry learning environment in accommodating different cognitive styles and how it enhances students' learning without any supporting evidence to their claim.

The claims made in the preceding paragraphs are however, not baseless. In the last two decades, Hoyles & Jones (1998), Kokol-Voljc (1999) and Laborde (2003) investigated and analysed dynamic geometry software and concluded that it enhanced the teaching and learning of traditional Euclidean geometry.

Arzarello et al. (2002), Falcade et al (2007), Gawlick (2002, 2005) found that the use of dynamic geometry software (DGS) promotes students' understanding of geometry. After analysing the interaction between geometry as a subject, Cabri as a computer tool and students as users, Straesser (2009) concluded that mathematical tasks involving the use of Cabri deeply change geometry and the process of learning mathematics. Falcade, et al., (2007) found that using the trace tool in Cabri provided learners with a variety of representations helping them to understand functional dependency in the context of functions. De Villiers (1997, 1998) cited in Hadas (2000) illustrated how students enriched investigations in dynamic geometry environments by asking the 'what if' questions, which lead to generalisations and discoveries.

On the negative, Holzl (2001) argues that dynamic geometry software has been reduced to a verifying approach, in which students are simply expected to vary geometric configurations in order to produce empirical confirmation of already formulated Euclidean results.

Most research on dynamic mathematics software I came across focus on technological and pedagogical issues, largely overlooking learners. Sharpe et al (2005) state that many studies have focused primarily on pedagogy and teachers' experiences rather than on how learners actually use and experience technology. Other studies have focused on the role of dynamic mathematics software as a tool in the process of learning mathematics. This research therefore identifies a gap on learners' experiences of using dynamic mathematics software (GeoGebra) to investigate a particular phenomenon (modulus functions).

There are other salient features of dynamic mathematics software which I considered to be important to this study. These are: provision for instant feedback; speed of execution; and ability to provide differentiated learning situations, which are discussed below in greater detail.

2.2.1 Instant feedback:

Dynamic mathematics software provides fast and reliable feedback, which is non-judgemental and impartial. It allows students to work with real-time images, something that cannot be done within traditional pen and paper environment. Clements (2000) argues that working in real-time encourages students to make their own conjectures and to test out and modify their ideas. This study, involved participants using a dynamic mathematics software (GeoGebra) to construct graphs of functions $y = f(x)$ to $y = |f(x)|$ or $y = f(|x|)$ and manipulating the representations using 'sliders', getting immediate feedback on the nature of the transformed representations. Instant feedback allowed participants the freedom to construct their own understanding of modulus functions based on the common features of the graphs that were immediately presented on the screen. It also allowed participants to monitor and manage their learning which is consistent with constructivist based learning.

2.2.2 Speed of execution:

Dynamic mathematics software enables students to produce many examples when exploring mathematical problems. This supports observation of patterns, fostering the making and justification of generalisations. Dick (1992) and Hopkins (1992) observed that dynamic mathematics software allowed students to explore more difficult problems, by providing quick ways of presenting data in different formats, something that was slow and tedious while using pencil and paper. By reducing time, that in the past was spent learning and performing tedious pencil and paper arithmetic and algebraic algorithms, dynamic mathematics software allows students to spend more time developing mathematical understanding, reasoning and application. It provides participants with learning tools that mediate their learning.

In this study, GeoGebra presented participants with opportunities to investigate many cases of modulus functions at high speed. One aspect that the study explored was whether this increase in the speed of execution promoted deeper understanding of the concept of modulus functions. The focus was on whether the increased speed of execution allowed participants to spend more time focusing on understanding things that change and things that stays the same, and the time to interpret results (Dick 1992, Hopkins 1992).

2.2.3 Differentiated learning:

DfES (2002a) report, notes that the use of dynamic geometry software facilitates independent learning. This assertion is supported by findings from Mehdiyev (2009), Ruthven et al (2008), Falcade et al (2007), Hoyles & Noss (2009), BECTa (2003, 2004), Hollebrands (2007), Ittigson & Zewe (2003), Gawlick (2002) and Oldknow & Taylor (2003) which show the impact of dynamic mathematics software on teaching and learning of mathematics.

The next section looks at the role of multiple representations in the learning of mathematics and how this impacts on this study.

2.3 The role of multiple representations on teaching and learning.

The role of GeoGebra in probing understanding of modulus functions through the use of its multiple representational facilities formed the foundation of this study. The use of multiple representations has been strongly connected with the complex process of learning in mathematics, and more particularly, with seeking students' better understanding of important mathematical concepts. Ozgun-Koca (2001) argues that the use of multiple representations has unavoidable contributions on meaningful algebra learning. Bayazit and Aksoy (2007) argue that representing the same concept in two different ways on the same screen promotes learners' depth of understanding and development of knowledge across the representations. This section focuses on the contributions of multiple representations in the learning and teaching of mathematics.

Ozgun-Koca (2001) defines multiple representations as mathematical embodiments of ideas and concepts that provide the same information in more than one form.

Duval (2006), classifies representations as either internal or external. Goldin & Janvier (1998) define external representations as structured situations that can be seen as embodying mathematical ideas. Cobb, Yackel and Wood (1992) view external representations as situated in the students' environments. On the other hand, constructivists view internal representations as those that exist inside students' heads. Internal representations describe some aspects of the process of mathematical thinking and problem solving (Cobb, Yackel and Wood 1992). They are individual cognitive configurations inferred from human behaviour (Goldin & Janvier 1998).

The need for students to use multiple representations has widely been accepted within mathematics education and research. This is manifest in the increased interest in designing free technology that supports multiple representations of mathematical concepts.

In 2001 Markus Hohenwarter initiated the development of GeoGebra with the aim of coming up with a completely new kind of tool for mathematics education in secondary schools. In 2011 DESMOS Inc. developed a free web-based platform for users to perform calculations, share and create interactive resources.

The important role that multiple representations play in the learning of mathematics can explain the wide number of investigations focusing on this topic in mathematics education research. A growing research base in cognitive psychology, cognitive science and mathematics education points to the crucial role of multiple representations in mathematical problem solving. Socio-cultural discourses have increasingly recognised multiple representations as central to the appropriation of knowledge through representational activity. The discourse on the nature and role of representational environments is well established, with several key texts devoted solely to this theme (Confrey, 1990; Janvier 1987, Kaput 1989).

Research in cognitive science and mathematics education has focused on the important role played by multiple representations in mathematics learning (Mehdiyev, 2009; Ruthven et al, 2008; Yerushalmy & Schwartz, 1999; Dufour-Janvier et al, 1987; Kaput 1989). Yerushalmy & Schwartz (1999), claim that multiple representations allow learners to use a rich set of both symbolic and graphical representations, hence building a richer and deeper understanding of mathematics concepts. Panasuk (2010) cites research studies (Ainsworth et al 2002, Lowrie, 2001 and Swafford & Langrall, 2000) that have focused on multiple representations and their subsequent impact on learning mathematical concepts. Stenning, Cox & Oberlander (1995) cited in Ainsworth et al (2002) gave learners graphical calculators and found that high performing students benefited from using graphical calculators while lower performing students preferred textual instructions.

Other research studies highlight the contributions of multiple representations to the teaching and learning of mathematics. Cordova & Lepper (1996), Cox et al. (2003) and Dikovic (2009) found that multiple representations positively affect pupils' attitudes and understanding mathematical concepts. Dufour-Janvier et al (1987), investigating the accessibility of representations concluded that multiple representations mitigate learning difficulties by making mathematics more attractive and interesting. Clark-Wilson (2010) and Duncan (2010) concluded that multiple representations stimulate investigations and help students develop their own learning skills. Kaput (1989), Skemp (1987), Hiebert and Carpenter (1992) argue that multiple representations help learners develop deeper

and more flexible understanding of mathematical concepts. Studies by Ozgun-Koca (2001) and Pitts (2003) provide evidence about the effectiveness of multiple representations based instruction in college algebra course. They found that the use of multiple representations enabled students to establish connections between varieties of representational modes.

Kaput (1992) points out that the use of multiple representations presents a clear and better picture of a concept or idea. Kaput argues that when multiple representations are utilised to illustrate various aspects of a mathematical idea, they contribute to the complexity of the learning environment and reduce the cognitive load on the part of learners. Learners, who seek deep understanding, are expected to grasp, not only the dynamic nature of each representation but also the dynamic connections among multiple representations (Kaput 1992). According to Duval (2006), the most difficult aspects of learning mathematics, is learning how to handle conversions between different representations of the same concept.

Schoenfeld, Smith and Arcavi (1993) cited in Ainsworth (1999) examined one student's understanding of mathematical functions using a multi-representational environment that exploited both algebraic and graphical representations to support learning. The student's increasingly successful performance led the researchers to conclude that she had mastered fundamental components of the learning domain by exploiting multiple representations involving algebra and graphs.

Findings from research studies conducted by Orton (1983) and Tall (1985) indicate that the use of multiple representations is advantageous to promoting conceptual understanding of graphs of derivatives.

In a non-computer environment, Brenner et al (1995) conducted a twenty day multiple representation study including variables in algebraic problem solving with seventh and eighth graders. In post-test results, they found that treatment students were more successful in representing and solving problems involving functions and words. Multiple representation-based instructions promoted conceptual understanding of algebra and made students conceptualise functions better than when they used pencil and paper.

Cikla (2004) investigated the effects of multiple representation-based instructions on Seventh Grade students' performance in algebra in a public school in Turkey. The conclusions from Cilka's study revealed that multiple representation-based instructions had a significant effect on students' algebra performance.

However, there are some noted weaknesses or disadvantages of using multiple representations in mathematics teaching and learning. Yerushalmy (1991) found that even after extensive experience with multi-representational learning experiences designed to teach understanding of functions, only twelve per cent of students gave answers that involved both numerical and visual representations. Most answers indicated that pupils used one representation and neglected the other. Yerushalmy's research suggests that appreciating the links, mainly across multiple representations, is not automatic.

Poppe (1993) explored the effects of differing technological approaches to calculus on students' use and understanding of multiple representations when solving problems. Poppe found that although students realised that tables, graphs and mapping diagrams were helpful, they did not use them simultaneously, in order to solve unfamiliar problems unless suggested to do so. Dufour-Janvier et al (1987) investigated the accessibility of representations and concluded that linking of the representations is sometimes abstract to students, and this could provoke lack of meaning to them.

Duval (2006) noted that when the roles of representations in source registers and representations in the target registers are inverted within a multiple representation conversion task, the problem is radically changed for students. This was noted when students could easily draw graphs from given equations but failed to present correct answers when the registers were inverted (i.e. starting with the graph and asking for the algebraic function that represents the graph).

Advocates of a constructivist approach to education argue that dynamically linking representations through use of technology leaves a learner too passive in the process (Ainsworth 1999). Ainsworth argues that such dynamic links discourage reflection on the nature of the transformation, leading to failure by the learner to construct the required

understanding. Students can see things dynamically changing but might not understand how they are changing and the role of the teacher in providing guidance and explanations is crucial.

The next section focuses on the features and interface of GeoGebra. GeoGebra has the capacity to link mathematics object to each other in a multiple representational environment. It provides real-time dynamic changes to all corresponding representations that are simultaneously displayed on the computer screen. GeoGebra provides options to move between visual mathematics options on its multiple screens, and options to dynamically change the objects. Images on the multiple screens can be enlarged for clarity, functions can be modified and colour coded. GeoGebra offers learners the possibility to simultaneously access different representations of the same concept.

2.4 CASE OF GEOGEBRA:

In choosing software appropriate for this study I considered four key aspects (pedagogical; mathematical; organisational and accessibility). The pedagogical aspect highlighted whether the software could be used to help participants learn modulus functions, develop concepts, increase knowledge, improve understanding and reinforce skills. The second, mathematical aspect, considered whether the software had facilities to process algebraic and graphical functions? Was it possible for the learners to link the different semiotic registers dynamically? Thirdly, it was important to consider the organisational aspect. This looked at whether the software could produce graphs easily and efficiently, manage time and provide multiple-representations of the same aspect.

Finally, I considered the aspect of accessibility and cost. Although computer software for mathematics has developed extensively, concerns have been raised about the apparent limited opportunities for students to access technology during mathematics lessons (Ofsted, 2008, BECTA, 2007). Software packages like Autograph and Geometer Sketchpad were, at the time of this study, available on the college portal but not accessible to students outside college hours (0830 – 1700 hours). During college hours, groups needed to book time in the Learning Resources Centre to access these packages. Sometimes the college network was very slow and made it difficult to plan a full lesson in the Learning Resources Centre. Most of the time was consumed while

logging in. Ofsted report (2008) and Miller et al. (2008) highlight difficulties in accessing computers for learners in mathematics lessons in schools in England as major impediments to integrating ICT into classroom practices.

The dynamic mathematics software that met all the set criteria for this study was GeoGebra. GeoGebra is an open-source software which was developed, specifically for teaching of mathematics in schools by Markus Hohenwarter and an international team of programmers. GeoGebra was designed to contribute to the achievement of specific tasks in the teaching and learning of mathematics. It is freely available and can be downloaded from www.geogebra.org.

Due to several useful features of the package, the GeoGebra community is growing, not only in England but internationally. The community involves international developers, mathematicians, mathematics educators, classroom teachers and students (Hohenwarter & Hohenwarter, 2009; Hohenwarter & Preiner, 2007). GeoGebra has evolved in response to the demands of mathematics itself.

GeoGebra has also gained growing international recognition in the field of research in mathematics education, since its official release in 2006. In the last decade, there has been a lot of focus on GeoGebra. Hohenwarter and Jones (2007), Hohenwarter and Preiner (2007), Hohenwarter & Lavicza (2007), Chrysanthou (2008), Mehdiyev (2009), Dikovic (2009), Preiner (2008), have focused on the use and development of GeoGebra in mathematics teaching and learning.

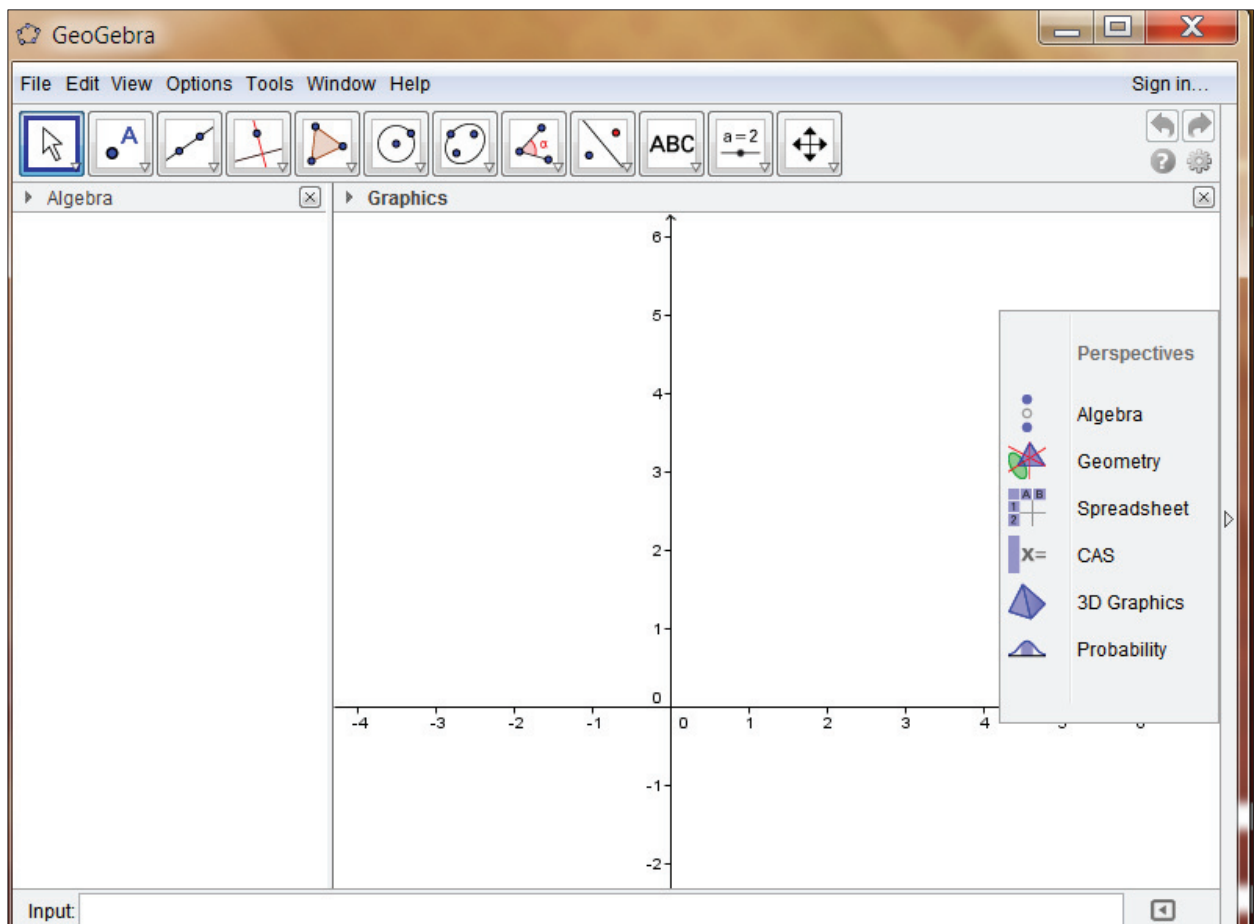
Some of the investigations (Hohenwarter and Jones 2007, Hohenwarter and Lavicza, 2007 and Hohenwarter and Hohenwarter 2009) focused much on developing GeoGebra as a powerful teaching resource, while a few (Ruthven & Hennessey 2003, Chrysanthou 2008, Mehdiyev 2009) have investigated the potential of GeoGebra to support mathematics activities which link algebraic and geometric reasoning.

Mainali and Key (2012) investigated teachers' impressions, feelings and beliefs concerning GeoGebra and found that all participants had positive impressions of using GeoGebra. Participating teachers seemed interested in the way they were engrossed in tasks and they expressed their enthusiasm in using the software. Dikovic (2009) looked

at applications of GeoGebra into teaching some topics in college level mathematics. Bayazit and Aksoy (2007) used GeoGebra to connect representations and mathematical ideas. Mehanovic (2011) investigated the potential and challenges of using of GeoGebra in Upper Secondary mathematics with students and teachers.

2.4.1 The GeoGebra interface:

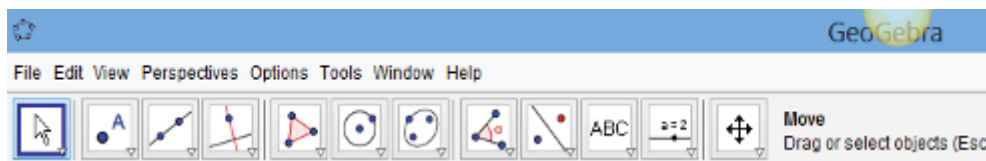
Fig 2.1: *The GeoGebra interface.*



GeoGebra combines ease to use perspectives of Algebra, Geometry, Spreadsheets, Computer Algebra Systems (CAS), 3-D and Probability (see Fig 2.1). GeoGebra is based on Java, an independent platform, which can run on most operating systems. It is open-source software, which enables students to create their own applets and upload them on YouTube for future reference or for the purposes of sharing with peers.

It is worth noting that the development and improvements in GeoGebra react to the needs of the users. There are certain features that were not available in GeoGebra 4.4 at the time of data collection (for example: automatic colour coding of representations of same concepts). GeoGebra Version 5.0 which was released after the data collection automatically assigns different colours to different functions entered on the same screen.

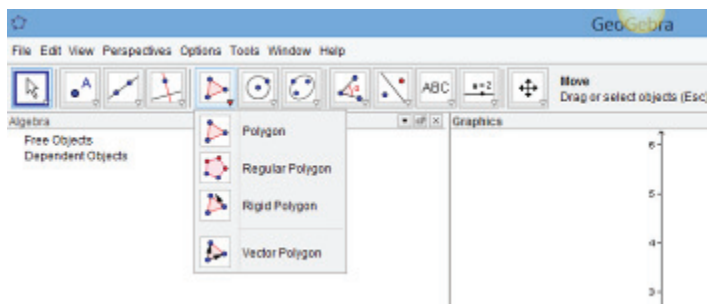
Fig 2.2: GeoGebra Menu:



Tools on the menu have been designed for

specific use. The toolbar consists of a set of toolboxes in which GeoGebra's dynamic geometry tools are organised. These tools can be activated and applied by using the mouse to achieve a specific task. Each icon on the toolbar consists of a drop-down menu that can be viewed by clicking on the icon (see Fig 2.3) allowing for selection of precise action.

Fig 2.3: A drop-down menu on one of the icons.



The drop-down menu allows students to select whatever option is applicable to their investigation. The name of the activated tool as well as the drop-down menu options, give students useful information on how

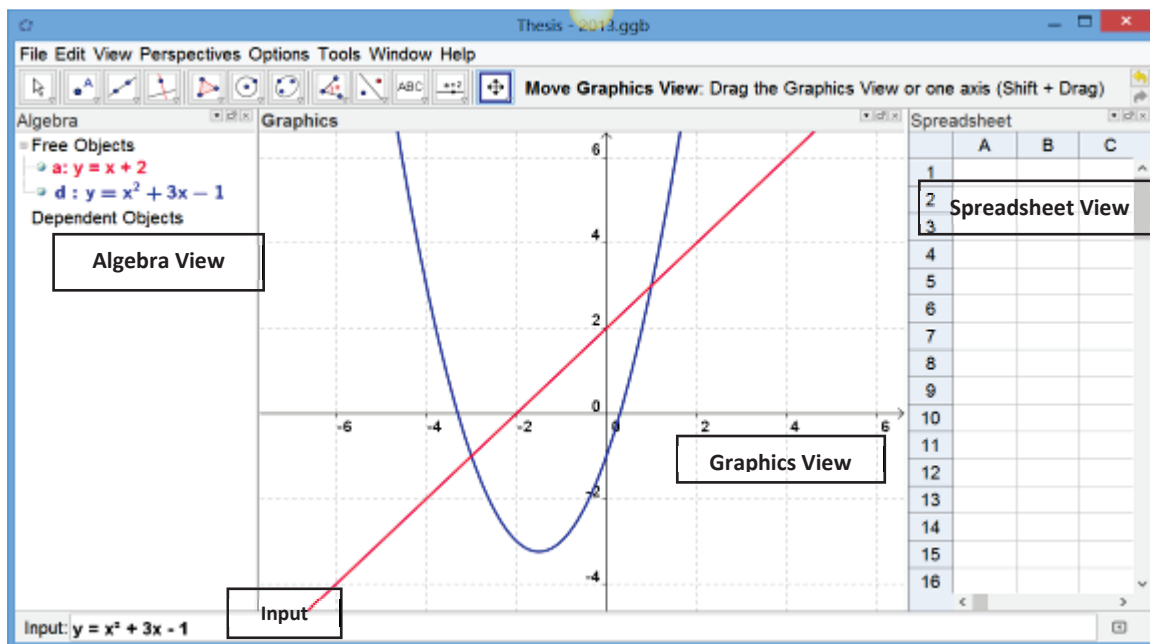
to operate the corresponding tool and how to create new objects.

GeoGebra is designed to contribute to the achievement of mathematical tasks. From the instrumental genesis perspective, it is an artefact that embeds knowledge and has characteristics implemented by mathematics educators and mathematicians (Rabardel 1995). Its development into a useful instrument can only be enacted by users as they work to achieve particular tasks. In this study, GeoGebra only becomes an instrument for the participants when they are able to use its functionalities to investigate modulus

function graphs. The instrumental approach advocates that schemes of use should be built by the participant.

GeoGebra provides a multiple screen presentation, which includes Algebra view; Graphics view; Spreadsheet view; Input and the Tool Bar at the top (see Fig 2.4). The base philosophy of the developers of GeoGebra has been to present the user with a large set of visualisation tools so that users can visualise abstract geometrical and algebraic concepts in real time. GeoGebra plays a role, not only in stimulating and shaping students' visual images but in providing access to new forms of representations as well as to multiple and linked representations (Kaput, 1986).

Fig 2.4: Multiple representations on the GeoGebra screen.



GeoGebra provides a versatile tool for visualising mathematical ideas (Hohenwarter 2006). Fig 2.4 shows the algebraic representation of $y = x^2 + 3x - 1$ and $y = x + 2$ in the algebra view and the corresponding graphical representation in the graphics view. Algebraic functions and their corresponding representations share the same colour for easy reference. The multiple screen representation facility provides an opportunity for students to see the multiple representations of the same mathematical concept.

Participants can also view and solve equations $y = x + 2$ and $y = x^2 + 3x - 1$ simultaneously. To solve simultaneous equations $y = x + 2$ and $y = x^2 + 3x - 1$, learners can select the intersection icon found in one of the drop-down menus. When they click on each graph in the *graphics view*, the coordinates of the points of intersection are automatically displayed on the *algebra view*, hence giving the required solution of the simultaneous equations.

GeoGebra screen has multiple windows (algebraic, graphical, input and tool bar) and provides facilities to manipulate representations in the different windows. It is possible for the users on one hand, to investigate the parameters of the equation of a curve by dragging the curve in the graphics window and observe the changes to the equation in the algebraic window. On the other hand, users can change the equation of the curve directly and observe the way the representation in the graphics window change (Hohenwarter and Jones 2007).

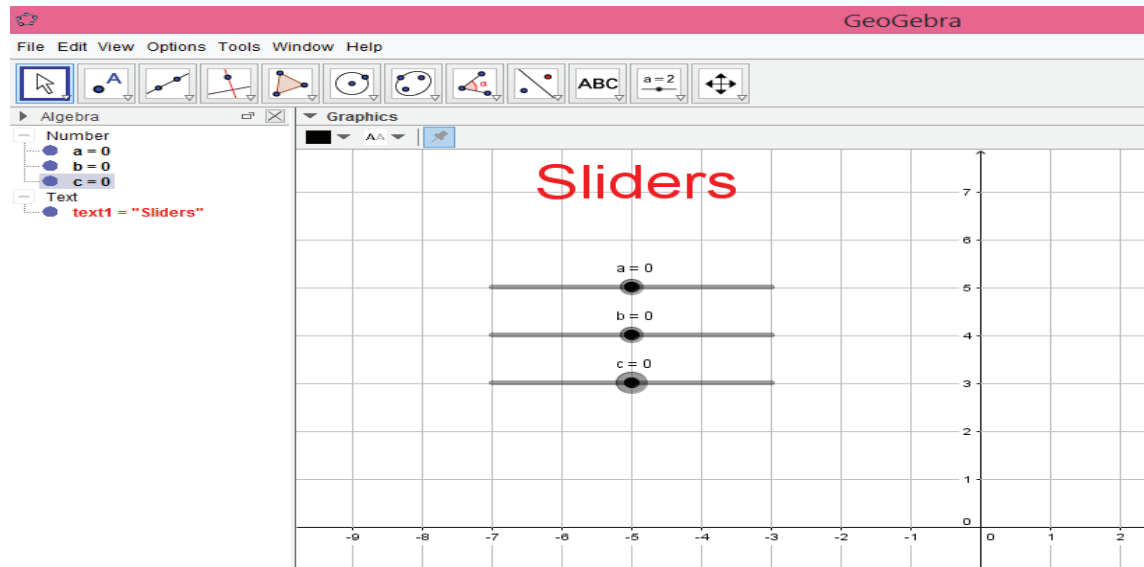
GeoGebra has multiple representations for every object: the algebra screen shows either, the coordinates, an explicit or implicit function, or an equation in parametric form, while the graphics screen displays the corresponding graphical representation. In GeoGebra, both representations can be influenced directly by the user. The graphical representation can be changed by dragging points on *sliders*, while the algebraic representation is changed dynamically. The algebraic representation can be modified using the keyboard and GeoGebra will automatically adjust the related graphical representation.

Sliders play an important role in the manipulation of multiple representations on the GeoGebra screen. The next section discusses sliders and their functions in the teaching and learning of mathematics.

2.4.2 Sliders:

The graphics register in GeoGebra has a slider component that forms an environment with dynamic variations. Variations in the graphics register are a key element in the construction of associated meaning of functions.

Fig 2.5: Sliders



Sliders play a crucial role in investigating graphs of functions that involve parameters. With the rise of dynamic mathematics learning environments, sliders are increasingly being used as pedagogical tools to create interactive mathematics activities that encourage students to explore mathematical ideas. Lingguo and Selcuk (2010) argue that sliders play a pedagogical role in the learning of concepts, where multiple representations are used in order to understand the phenomenon. From a pedagogical perspective, sliders serve various roles in mathematical teaching and learning, providing opportunities and challenges for learners. In the case of graphs of modulus functions $y = |ax^2 + bx + c|$, sliders are used for the underlying values of a , b and c in order to draw different representations. It is worth noting that sliders are used to set up initial conditions for a problem situation with the explicit intention to allow for participants' guided investigations.

Sliders are used for dragging, enabling participants to manipulate initial conditions of a problem situation and observe dynamic changes in a variety of related multiple representations. It allows the user to adjust the values of the corresponding variable through dragging. It is easy to use sliders in a function of the format $y = ax^2 + bx + c$ to avoid typing varying values of a , b and c repeatedly. Sliders enable learners to explore multiple cases of functions, without changing the degree of polynomial.

2.4.3 The uniqueness of GeoGebra:

What defines GeoGebra as a unique piece of software is its semiotic abilities, for instance in the simultaneous and dynamic representations of mathematical concepts on the same screen. This was more interesting than the software's capability to solve specific problems in an effective way. The availability of GeoGebra in and outside the classroom offers new opportunities for students to communicate and analyse their mathematical thinking.

GeoGebra provides constant double representations of mathematical concepts. The conversion from one form of representation to another could cognitively differ in a GeoGebra environment, compared to pencil and paper environment (Misfeldt 2008). According to Duval (2006), the most difficult aspects of learning mathematics, is learning how to handle conversions between different representations of the same concept. The constant presence of dual representations on a GeoGebra screen marks a potentially large difference in accessibility to the mathematical topic of modulus function graphs.

GeoGebra extends the concepts of dynamic geometry to the fields of algebra and mathematical analysis. It fosters justification and generalisation by enabling fast, accurate sketches and exploration of multiple representational forms.

Dikovic (2009), Hohenwarter and Preiner (2008) argue that GeoGebra is more user-friendly compared to a graphic calculator, and that it offers an easy-to-use interface, multilingual menus, commands and a help line.

2.5 THE CASE OF MODULUS FUNCTIONS:

Applications in the algebraic domain of graphs, equations and inequalities of modulus functions are an integral part of the Core 3 GCE mathematics curriculum in England. The Core 3 Mathematics specifications (2014 – 2015) for Edexcel specify that students should be able to understand:

- (i) The meaning of the modulus notation and modulus function $|x|$;
- (ii) Algebraic solutions of equations and inequalities involving modulus functions
- (iii) Sketch the graphs of $y = f(x)$; $y = |f(x)|$ and
- (iv) $y = f(|x|)$ given the graph of $y = f(x)$

This study focused on modulus functions. I selected the concept of modulus functions because of the investigatory nature of the concept when solving equations and inequalities and the predominant role graphical representations can play in understanding the concept.

I came across only one research literature on students' understanding of absolute value functions by Monaghan and Ozmantar (2006), and virtually nothing on the use of GeoGebra on modulus function (or absolute value functions). Most of the literature I came across was more or less equally divided between reports in professional journals for teachers and short research reports about the potentiality of GeoGebra and its use in teacher education or university calculus.

The selection of the topic 'modulus functions' emanated from my observations when teaching the modulus functions topic. I noted students having problems in dealing with questions involving modulus functions. A few of the problems are highlighted in the three extracts presented below.

The first question required students to solve the equation $|2x + 1| = 3x - 2$. One student presented:

Insert 2.1

$$\begin{aligned}
 |2x + 1| &= 3x - 2 \\
 \text{If } |2x + 1| &= 2x + 1 \\
 2x + 1 &= 3x - 2 \\
 3 &= x \\
 \text{If } |2x + 1| &= -2x - 1 \\
 -2x - 1 &= 3x - 2 \\
 1 &= 5x \\
 x &= \frac{1}{5}
 \end{aligned}$$

The student's work was presented in a very logical way, considering the two possible cases when $|2x + 1| = 2x + 1$ and when $|2x + 1| = -2x - 1$. Without the aid of a diagram the student failed to recognise that the functions $f(x) = |2x + 1|$ and $g(x) = 3x - 2$ only have one point of intersection (3, 7).

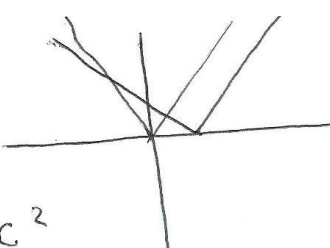
Another question required students to solve $|3 - 2x| = 4|x|$. One student presented this solution (*my own typing*).

$$\begin{aligned}
 |3 - 2x| &= 4|x| \\
 3 &= 6x \\
 x &= \left| \frac{3}{6} \right|
 \end{aligned}$$

What attracted my attention to this solution was the use of the modulus function notation in the final answer, which is already positive. This highlights a misunderstanding of the absolute value concept.

Another student used the aid of a sketch to solve the same problem as above, but abandoned the graph for an algebraic method.

Insert 2.2

$$\begin{aligned} |3-2x| &= 4|x| \\ (3-2x)^2 &= 16x^2 \\ (3-2x)(3-2x) &= 16x^2 \\ 9-12x+4x^2 &= 16x^2 \\ 4x^2+4x-3 &= 0 \\ x &= \frac{-4 \pm \sqrt{(4)^2 - 4(4)(-3)}}{8} \\ x &= \frac{1}{2} \text{ or } x = -\frac{3}{2} \end{aligned}$$


The three selected examples displayed the varying approaches learners used to solve equations involving modulus functions. The first two examples used algebraic methods with no link to graphical representations. An attempt to use graphical representations was made in the third extract, but was abandoned in favour of the algebraic method of squaring both sides of the equation.

These observations are not unique to these three students only. The MEI examiners' reports on questions involving modulus functions also made similar observations:

The June 2008, Question 2 examination report noted that, in solving the inequality $|2x - 1| \leq 3$ most candidates correctly derived $x \leq 2$, but had difficulties in handling the left hand boundary of the inequality. Some candidates squared both sides, getting $x^2 - x - 2 \leq 0$, but made the error of arguing that $(x+1)(x-2) \leq 0 \Rightarrow x \leq -1$ and $x \leq 2$. Very few students used a graphical approach.

The June 2012 report identified a very common error where candidates were failing to change the inequality sign when proceeding from $-2x > 5$ to $x < -2.5$. It also noted that a few candidates sketched the graph of $y = |2x + 1|$ to solve the problem. Other errors seen occasionally were $|2x + 1| > -4$ when the modulus

function $|2x + 1|$ cannot be negative. Some candidates did not seem to appreciate that nesting the two inequalities $-2.5 > x > 1.5$ is incorrect. (MEI Core 3 June 2012, Question 2).

In solving the equation $|2x - 1| = |x|$, the report noted that most candidates attempted this by considering $\pm (2x - 1) = \pm x$, with some thinking that this led to four different possibilities and indeed finding more than two solutions by faulty algebra. A few candidates squared both sides, found the correct quadratic and solved this by either factorising or formula. Very few candidates used the graphical method (MEI Core 3 June 2011, Question 1).

The reports highlight problems that students face when solving equations involving modulus functions. There is very little use of graphs in solving equations and inequalities involving modulus functions. MEI curriculum specification (2015) recommends the use graphs when solving equations such as $|x - 2| = 2x - 1$ or inequalities like $|3x + 2| < 8$ or $|x - 3| \geq 5$ to avoid getting superfluous solutions from algebra-only based solutions.

The examination reports and students' work show that the concept of modulus functions is difficult for students to understand. A study carried out by Monaghan and Ozmantar (2006) in Turkish schools also established that students became confused about the difference between the graphs of $y = |f(x)|$ and $y = f(|x|)$. Monaghan and Ozmantar (2006) also cited the problems encountered in the introduction of the formal definitions of $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$, as creating difficulties for students who see this as "contradictory to their perception of an absolute value as a positive number" (Monaghan and Ozmantar 2006:236).

2.6 SUMMARY:

The present study investigated the use GeoGebra to teach modulus functions to six 'A' level students. This chapter provided an overview of literature pertaining to the development of a theoretical framework for this study, while at the same time taking into account the theories behind multiple representations. The review presented a number of key research findings on the use of multiple representations. The review attempted to

get a clearer understanding of the ways learners used the multi-representational technologies to learn mathematical concepts.

Review of related literature focused on the theoretical framework and the philosophical paradigms chosen for this study. It looked at multiple representations in general and GeoGebra in particular and how technology has / is impacting of learners' understanding of concepts. The focus topic, modulus functions, was also discussed highlighting its structure and the rationale of choosing it. Issues considered in the review of literature helped in focusing the study on the use of technology in teaching and learning mathematics.

CHAPTER 3:

3.0 METHODOLOGY, DESIGN and DATA COLLECTION and ANALYSIS:

There is a range of methodologies and methods available in mathematics education. In this chapter I make explicit the theories that influenced my work, since these theories influenced the ways in which I collected and analysed data. The chapter describes the methodology applied to answer the research questions. Specific topics included in this chapter are: research methodology, research design, participants and sampling techniques, methods of data collection, trustworthiness of the study, methods of data analysis and ethical considerations.

3.1 RESEARCH METHODOLOGY:

Crotty (1998) refers to research methodology as a systematic and theoretical analysis of the methods applied in a field of study. In education, research is situated within diverse epistemological traditions (i.e. positivism, constructivism, pragmatism, postmodernism), creating a diverse intellectual and methodological landscape.

Crotty (1998) defines epistemology as the study of knowledge and how we know what we know. Maynard (1994) cited in Crotty (1998), and Gray (2009), define epistemology as a philosophical background used to decide what kinds of knowledge are legitimate or adequate for a particular research. An epistemology is concerned with what knowledge is and how it can be acquired, and the extent to which the acquired knowledge is important to any given subject.

There are two main methodological positions namely qualitative and quantitative, used in education research. Quantitative methods are mostly employed by positivists (Crotty 1998). Quantitative research is based on measurement of quantity or amount. The process is described in terms of one or more quantities. The results of quantitative research are essentially a number or a set of numbers. Quantitative research adopts two major epistemological positions: the first position is the assumption that it is possible to acquire knowledge about the world unmediated and with no interferences. This implies that objectivity is possible, because everyone observes things in the same way. The

second assumption is that observation is never objective but always affected by the social constructions of reality.

Rajasekar et al (2006:9) list the characteristics of quantitative methods as: “numerical; non-descriptive; applies statistics or mathematics and uses numbers; an iterative process whereby evidence is evaluated; conclusive; and the results are often presented in tables and graphs”. The great advantage of quantitative approach, derived from these characteristics, is that the data is usually easy to replicate.

On the other hand, qualitative research is usually employed by constructivists. Qualitative research follows a naturalistic paradigm based on the notion that reality is not predetermined but constructed by research participants (Polit 2001).

As with all research in education, there are different interpretations and definitions for qualitative research. It is alternatively called naturalistic inquiry, field study, case study, participant observation and ethnography (Bryman, 2008; Merriam 1998; Yin 2009). Holloway and Wheeler (2010) refer to qualitative research as a form of social enquiry that focuses on the way people interpret and make sense of their experiences. Burns and Grove (2003:19) describe a qualitative approach as “a systematic approach used to describe life experiences and situations to give them meaning”. According to Creswell (2008) qualitative researchers keep a focus on the meanings that the participants hold about the problem or issue. In a mathematics classroom, data is collected as participants interact in small groups; in whole-class discussions; with the teacher or working individually.

This study follows an Interpretive Phenomenological Analysis (IPA) approach to gain insight into participants’ experiences while using technology. Interpretive phenomenological analysis is a recently developed and rapidly growing approach to qualitative enquiry (Smith et al 2012). It originated and is best known in psychology but is increasingly being picked up by researchers working in cognitive psychology, social and health sciences. IPA focuses on experiences of individual participants, looking at what sense they make of what is happening to them. The founding principal of IPA, which has its roots in the field of healthcare research, is that experience should be

examined in the way that it occurs and in its own terms. IPA is usually used to study small numbers of participants aiming to reveal the experience of each individual. It is concerned with the detailed examination of lived experience.

This study focused on the experience of six participants in a classroom environment, using GeoGebra to investigate modulus functions within a constructivist paradigm. The rationale for using IPA in this research was to explore and describe how learners experienced technology in the process of investigating modulus functions. The choice of an IPA approach seemed appropriate since the study is concerned with exploring participants' experiences. The classroom setting remained the direct source of data (Fraenkel & Wallen, 2003). An IPA approach was also appropriate to capture real-time experiences as learners were at work.

3.2 RESEARCH DESIGN:

Parahoo (1997) describes a research design as a plan that describes how, when and where data is collected and analysed. Philliber et al. (1980) consider a research design as a blueprint for research, dealing with at least four problems: research questions; which data are relevant; what data to collect and how to analyse the results. It also includes how data is collected, what instruments are employed, how the instruments are used and how the collected data is analysed. For Borg and Gall (1989) a research design is a process of creating an empirical test to support or refute a knowledge claim. Polit (2001:167) define a research design as "the researchers' overall, for answering research questions or testing the research hypothesis". Creswell (2008) views research designs as specific procedures involved in the last three steps of the research process: data collection, data analysis and report writing. All the above definitions (Creswell 2008; Parahoo1997; Borg and Gall 1989; Polit 2001 and Philliber 1980) share a common view that research designs lay out a process for data collection and data analysis.

The research design selected for this study incorporated IPA theories within a socio-constructivists paradigm, in an attempt to explore how participants interacted with technology in the mathematical activities set for this study. This study adopted a case study approach, structured according to Crotty's (1998) suggested research process. Crotty (1998:4) suggests that in developing a research design, the researcher should

answer basic questions: “what theoretical perspective lies behind the methodology; what methodology controls our choice and use of methods and what methods are proposed to be used”.

The IPA approach, grounded within a constructivist epistemology, considers truth and meaning as constructed and interpreted by individuals through experiences. It is concerned with the detailed examination of human lived experience expressed in its own terms rather than according to predefined category systems.

Since this study was concerned with the process rather than the outcomes or product, the research questions were best answered through a case study design within an IPA paradigm. Yildirim & Simsek (2006) define a case study as an empirical research method which studies a phenomenon within its real life framework, in which boundaries between the fact and the content are not clear. It is used when more than one evidence or source of data is available. In case studies, the researcher does not only observe the subject under research as in quantitative designs, but participates in the study in person to analyse both the subject and the participants.

My focus on a small group of participants, working with GeoGebra to understand a specific phenomenon suited a case study approach. I collected data through screencast video recordings and compiled field notes. The IPA approach allowed me to get in-depth understanding of the experiences and perceptions of participants as they used GeoGebra to explore the concept of modulus functions and how they developed an understanding of the concept.

3.2.1 The Pilot Study

A pilot study was carried out in October 2013, with different participants who were also working at Core 3 'A' level mathematics. The pilot study was set out to test the research instruments, the software GeoGebra and viability of group size. It also looked at the suitability of screencast video recording software and noted its shortcomings. The free version of the screencast video recording software (JING) allowed five minute recording sessions and participants had to keep switching it on after every five minutes for more recordings. Sometimes, when immersed in the problem, participants failed to realise that

recording had stopped. After noting these shortcomings, I paid a subscription to get a full version of JING, which proved to be a better version than the free version used during the pilot study. The new version allowed for longer periods of recording and provided timeframes in which recordings took place.

The pilot study helped me to review my research questions to focus on two aspects: (i) students' experiences with the software GeoGebra as an instrument in their environment and (ii) how the use of multiple representations contributed to the learning of the concept of modulus functions. These two aspects influenced the choice of the theoretical frameworks and sampling techniques for this study.

3.2.2 Sampling techniques, sample and participants:

Sampling involves the selection of a number of study units from a defined study population. Fraenkel & Wallen (2006) define a sample as a small group drawn from a population on which information is obtained. Qualitative researchers usually work with small numbers.

Sampling is a process of selecting a group of people with whom to conduct a study (Burns and Grove 2003). The most common sampling methods used in qualitative research are purposive sampling, quota sampling and snowballing. IPA research usually works with purposive sampling because it seeks to obtain insights into particular experiences that occur within a specific location, context and time (Smith et al 2012 & Gray 2009). According to Parahoo (1997), in selecting a sample, researchers use their own judgement to select participants to be included in the study based on their knowledge of the phenomenon. Burns (2000), Miles & Huberman (1994), cited in Gray (2009:182), and Creswell (2008) advise that the best strategy is to initially target those cases that are most likely to yield the richest data or in-depth picture. Smith et al (2012) argues that participants should be selected on the basis that they represent a perspective rather than a population.

3.2.2.1 Sample:

This study was carried out in a Sixth Form College located in a City in England. The college serves approximately 1000 students, 150 of who study mathematics at AS, A2 and Further Mathematics Level. Sampling was theoretically consistent with the qualitative paradigm in general, and with IPA's orientation in particular. The research sample was obtained from a group of 'A' level mathematics students, who volunteered to take part in the research. The topic under investigation was peculiar to a particular group of A Level students, therefore participant selection process considered two criteria. (1) All participants were drawn from volunteers studying GCE Core 3 mathematics module. (2) Participants were also supposed to be available during scheduled workshop sessions. This was to allow continuity and progression in activities. The workshop sessions are slots reserved on the college timetable, outside the normal mathematics timetable. The sessions are intended to offer students an opportunity to get extra help outside classroom time but during the college day.

Due to detailed case-by-case analysis of individual transcripts, IPA studies usually benefit from a concentrated focus on a small number of cases. Smith et al (2012) do not set a limit on the number of cases to be studied at doctoral level research, but however proposes a small number of between four and eight cases. I settled for a sample of six 'A' Level participants, selected from classes on my timetable, to develop an insight into: a) how they engaged GeoGebra in their learning of mathematics and (b) how GeoGebra, through its multiple representational facilities, related to and contributed to their whole learning experience?

With permission from the college head and the curriculum leader, I explained my research project and its objectives to all students in my A2 classes from whom I requested for volunteers. Initially eight students volunteered but two withdrew due to clashes on their timetable. Six students took part in the project. I made it clear that the project would be contacted outside their normal timetable. The six participants met as a small group during scheduled workshop sessions under my supervision and worked mostly in pairs using laptops. A sample size of six was also consistent with Holloway and Wheeler (2002:128) who assert that "sample size does not influence the importance

or quality of the study” and note that there are no guidelines in determining sample size in qualitative research. Onwuegbuzie & Leech (2007) suggest that sample sizes in qualitative research should not be so large that it becomes difficult to extract thick, rich data. On the other hand, Flick (2006) suggests that the sample should not be too small so that it becomes difficult to achieve data saturation.

The selected participants had not encountered any work on modulus functions in Core 1 or Core 2 modules. At the time of the study, they had been introduced to the concept of functions and had covered work involving definition of functions, types of functions (even, odd or periodic), domains, ranges, composition of functions and inverse functions and their graphs, including graphs of functions and their inverses. From the Core 1 and Core 2 modules, participants had encountered curve sketching of simple polynomials by identifying x and y intercepts and finding coordinates of the maximum or minimum point (Core 1) and by identifying turning points using calculus (Core 2).

3.2.2.2 Participant profiles:

The names used in these profiles are pseudonyms, for confidentiality, but the profiles are real.

John: John was 17 years old at the time of this study. He had passed AS mathematics with an overall grade C. In the initial interview, John had indicated that he had struggled with curve sketching in Core 1. However, he pointed out that he enjoyed curve sketching when calculus (Core 2) was used to identify stationary points. College records indicated that John had done well in his Core 1 and 2 modules averaging a score of seventy five per cent in both modules but had scored a grade E in the Statistics 1 module. On the functions topic, John indicated that he was still struggling to distinguish between even, odd and periodic functions.

Peter: Peter was 18 years old. He had attained an overall grade B in his AS mathematics and was predicted to get a grade B in ‘A’ level mathematics. Peter was also studying Further Pure mathematics (FP 1) module and he was confident with his curve sketching including functions involving rational algebraic fractions, since this work is also covered in more detail in FP 1. Unlike John, classroom discussions and topic

assessment results indicated that Peter had mastered the aspects of functions covered by the time of the study (i.e. types of functions, inverse functions and curve sketching). From my own observations, Peter had no hesitations when it came to participating during lesson discussions.

Susan: Susan was 17 years old. She had attained an overall grade C in her AS mathematics and indicated that she needed some extra help in Core 3 mathematics. She indicated that she was fine with her mathematics during lessons but struggled to apply basic concepts when solving examination related questions. She came across as someone who lacked confidence and most of the time sought for second opinion from friends before answering a question.

James: James was 19 years old. He was on an intensive one year course covering AS and A2 mathematics. James had just joined the college and was doing all six 'A' level modules in one year. By the time of this study, James was still trying to come to grips with algebra at Core 1 and Core 2 level. He had been out of formal education for three years. James had already been identified by the department as a student who needed extra help. He lacked confidence and preferred to work on his own most of the time.

Emma: Emma was 18 years old. This was Emma's third year in college having repeated AS mathematics. She had attained an overall grade C on the second attempt of her AS mathematics. Emma claimed that she was comfortable with sketching graphs of algebraic functions from her Core 1 & 2 modules. She was still trying to figure out composition of functions and the types of functions (i.e. odd, even and periodic). Mid-topic assessment also indicated that she could find the inverse of a function but struggled to sketch the graph of the inverse function. Emma did not come across as a shy person. She enjoyed discussions.

Sophie: Sophie was 17 years old. Very confident and predicted a grade A in her A level mathematics. Sophie had attained an overall grade A in her AS mathematics. She enjoyed working on her own most of the time. Sophie had no problems with sketching graphs of algebraic functions. She could recognise links between topics previously learnt and current work. She came across as someone who could think outside the box.

None of the six participants had encountered modulus functions beyond its use in stating the validity of a binomial expansion as $|x| < 1$. The whole concept of graphs, equations and inequalities involving modulus functions was new to them. All had encountered the software GeoGebra while doing course work on numerical analysis in the Core 3 module a few weeks prior to the start of the research sessions. However, there were some aspects of GeoGebra pertinent to the learning of modulus functions that had to be introduced to them at the start of this study.

3.3 RESEARCH METHODS:

This section describes not only the research methods, but also considers the logic behind the methods used in the context of this study and explains why I used a particular method or technique and why I did not use other methods.

Gray (2009), Cohen et al (2003), Crotty (1998) refer to research methods as the range of techniques or procedures used in educational research to gather and analyse data related to the research questions. They are concerned with the collection of data, how data is analysed and validated. In this study, I used qualitative methods consistent with the IPA approach to gather data. Since the aim of this study was to investigate participants' experiences with GeoGebra in a specific classroom environment, I employed data collection methods that draw on ethnographic techniques such as case study, lesson observations and interviews. Participants were granted opportunities to work freely and reflectively and to develop and express their ideas independently.

3.4 METHODS OF DATA COLLECTION:

Consistent with the ontological and epistemological position that the world is socially constructed and all knowledge that we can have about it is subject to interpretation, IPA use interviews, focus groups, observations and other qualitative methods to get an in-depth sight into a field, with a richness of description not obtainable by quantitative research. Photographs, films and video are used as forms and sources of data (Flick 1998).

Data collection techniques chosen for this study were based on case studies within an IPA perspective. In case study research, data collection draws on multiple sources of

information such as observations, interviews and audio materials (Creswell 2008). This study used observations, video recordings and interviews. I collected data from participants' screencast and audio recordings while they investigated modulus functions. I also compiled field notes during the research sessions.

3.4.1 Screencast video and audio recordings:

To understand how participants used GeoGebra in the learning process, I used a screencast capturing software called JING. JING captures every activity on the laptop/computer screen including the entire desktop, menus and screencast video with sound from external sources, e.g. participants' voices. JING captured actual conversations and all the on-screen activities as participants were investigating modulus functions. By using JING, I was able to view participants' activities and listen to their discussions long after the lessons were finished. From a research perspective, the recordings from JING gave information about what tools in GeoGebra were utilised during the lesson activities and what approaches were used to solve a particular problem.

The use of GeoGebra and JING was more prevalent during Sessions I, II and III but appeared to fade away in the subsequent sessions as participants became more familiar with graphs of modulus functions and they could sketch them freehand. In the last three sessions I relied more on observations and field notes taken during session activities and discussions.

Pirie (1996) cited in Powell et al. (2003) observes that videotaping a classroom phenomenon is likely to be the least intrusive, yet most inclusive way to study the phenomenon. JING captured all on-screen activities highlighting a variety of approaches employed by participants while investigating modulus functions. The moment-by-moment screencast recordings from JING gave me an opportunity to re-examine data again and again. Recording sessions freed me from taking detailed notes on participants' activities and discussions, but allowed me more time to focus on group discussions as I moved from group to group taking notes.

This study took place over a three-week period in October 2014. Six sessions were conducted over a period of three weeks. Each session was one and half hours long, which is the duration of a normal timetabled lesson in the College where the study was undertaken.

Participants worked in pairs and were reminded to switch on the recording facility at the beginning of each activity. The first session was divided into three activities; with the first activity focusing on the introduction of GeoGebra and the concept of modulus functions. In the second activity, participants used GeoGebra to investigate graphs of linear functions involving modulus functions (See Appendix B). This was followed up by the third activity where participants continued to use GeoGebra to identify functions of given graphs involving modulus functions (see Appendix E).

During the second session participants investigated graphs of polynomials involving modulus functions using algebra. Data was collected from discussions through field notes and participants' handwritten work. In instances where they used GeoGebra to check their solutions, participants were encouraged to record their activities. Participants used GeoGebra in the third session to investigate and compare graphs of $y = |f(x)|$ and $y = f(|x|)$. They also used GeoGebra to investigate graphical representations of the sum of two modulus functions $|ax + b| + |cx + d|$. Activities in this session were captured on JING and these were complimented with field notes.

The fourth session focused on solving equations involving modulus functions. Some questions were selected from a worksheet (see Appendix B). Participants had the freedom to use GeoGebra or pencil and paper. Data was drawn from participants' answers to the exercise and screencast videos recorded during the session.

Session 5 focused on solving equations and inequalities involving modulus functions with the aid of graphs. The use of algebraic methods to solve equations and inequalities were discussed and the merits and demerits of using both methods were highlighted.

Research by Arzarello & Edwards (2005) has shown that children benefit from gesturing during mathematics instruction. One weakness of JING was its inability to capture gestures, such as finger pointing to the screen, facial expressions and body language

as participants worked in a GeoGebra environment. Some of participants' experiences expressed through gestures were therefore not captured. However, the group was small enough to enable me to recognise voices and identify which clip belonged to which pair in most instances.

3.4.2 Session observations:

Creswell (2008) views observation as a process of gathering open-ended, first-hand information as it occurs in a setting. Observations can be structured or unstructured, participant or non-participant (Bell, 2006). In unstructured observations, participants are prepared to spend a great deal of time on fieldwork, familiarising and accumulating data from which they anticipate that focus and structure will emerge.

I selected participant observation for this study. Bell (2006:186) says participant observation involves the researcher "participating in the daily life of an individual, group or community, listening, observing, questioning and understanding the life of the individuals concerned". I worked with a group of participants who were well known to me and I was carrying out a study in an environment very familiar to me, therefore it was difficult to stand back and not participate in some of the discussions (Robson 2009, Simpson et al 2003).

Observations during the sessions afforded me a chance to gather live data from session activities as it emerged, getting close to pairs and observing directly what was happening. The information obtained related to what was currently happening and was not complicated by participants' previous knowledge or future intentions. Observations also gave me direct access to the learning environment. I designed an observation protocol, to provide guidelines for field-note taking. This provided a list of issues I was looking for. I responded and took notes whenever an event on this list occurred. The notes compiled during observations complimented data collected from the JING screencast video clips.

Observation Protocol:

1. How participants selected the relevant windows and began to use GeoGebra for a specific task?
2. How participants navigated their own way through specific tools on the GeoGebra menu? Finding the correct syntax; identifying correct tools and movement of cursor between windows
3. Were participants able to complete activities within the allocated time frame?
4. How often did participants ask me for help?
5. Did participants ask peers for additional help instead of me?
6. Did participants link their investigations to previous knowledge?
7. Were there any emerging themes linked to but outside the confines of curriculum specification?

3.4.3 Interviews

Cohen et al. (2003) regard an interview as an exchange of views between two or more people on a topic. Gray (2009) defines an interview as a conversation between two people, which begins with the interviewer, with the purpose of collecting data relevant to their research, and focuses on content which is determined by the research goals. Creswell (2008) lists four interview approaches used in qualitative research: one-to-one, focus groups, telephone and electronic mail (e-mail).

Based on evidence from video recordings, I held group interviews with participants at the beginning of each lesson, to clarify or probe more on issues I had noted as 'unclear' or 'requiring further investigation'. Open questions were used to allow participants to best voice their experiences without being constrained by my perspectives. These open questions were based on screencast video clips viewed before the next session. In the interviews, I also asked participants to elaborate on their perception of GeoGebra with respect to the learning of modulus functions.

The following questions were used to guide the interviews:

1. Can you tell me something about how you used GeoGebra during the activities?
2. Did you find it easy to find the correct commands and tools? Which features did you find easy/difficult to use?
3. Did you enjoy using GeoGebra? Why?
4. Is there anything you found difficult or you could not do on GeoGebra during the activities?
5. How much do you feel the use of GeoGebra has helped you in this session to understand graphs of modulus functions, to solve equations and inequalities involving modulus function?
6. Do you feel GeoGebra has improved your understanding of modulus functions and made your learning easier?
7. Is there anything else you would like to share?

Data collected through group interviews was used to cross-check the accuracy of data gathered from screencast video clips and lesson observations (LeCompte et al 1982). Interviews provided a means of triangulating inferences made about graphs of modulus functions during the investigations.

A group discussion was held in the last session to find out participants' general experience with the software and check whether it had helped them to understand modulus functions. Aspects like ease of use, speed of execution and courage to attempt more complicated questions were asked to help answer the second research question focusing on participants' overall experience with GeoGebra. I listed participants' responses on a flip chart as we were discussing. This helped in controlling the speed of the conversation and allowed me to capture as much as was being discussed (see Appendix J for the list of responses)

3.5 METHODS OF DATA ANALYSIS:

I adopted an Interpretive Phenomenological Analysis approach to analyse the data. Data analysis started with organisation and description of the data modelled around an interpretive phenomenological analysis framework (Smith et al 2012). Initially, the activities in the learning arrangements served as the units of analysis. Line numbers were used to organise and document data, which is presented in excerpts of conversations and brief narratives of screencast video activities (Vignettes). Events in screencast videos and audios that related to research questions were transcribed and discussed. I constructed story lines as a reconstruction of participants' learning process and in some cases, modified terms used in the conversations, to provide clarity to the narratives.

The first research question, "In what ways did participants employ GeoGebra to support their understanding of the concept of modulus functions?" was answered, first by analysing the ways in which participants made meaning of the tools in GeoGebra. In the recorded sessions, I was looking for instances when participants related different graphical and algebraic representation registers to each other. I also looked at how participants used immediate feedback from GeoGebra to further explore the concept of modulus functions.

After careful observation of sessions, audio recordings and transcribing all video and interview data, an interpretive phenomenological analysis framework was used to analyse the data collected for this study. Data obtained was summarised and interpreted according to research questions. Quotations were included in the vignettes to reflect the opinions and experiences of the participants.

Audio and screencast video recordings, accompanied with participants' written work were converted to text in order to observe the epistemic actions of recognising, building-with and sketching the modulus function graphs. The analysis reflected the different representations of modulus functions and the type of GeoGebra tools used. The observation notes taken during the sessions were evaluated, compared and contrasted with video recordings. Finally, comments were made based on data, to interpret findings and explain the relations in between video clips and draw conclusions

3.5.1 Data Transcription:

Screencast video recordings from the lessons were transcribed throughout the investigation period. Transcribing is the process of transferring to a page the activities and positioning the discussions that occur during the recorded session (Powell et al. 2003). Transcription involved a close look at the screencast videos through repeated viewing, then recording them in text form. The transcription of recordings was also modelled around interpretive phenomenological analysis framework (IPA) also borrowing from some steps proposed by Powell et al (2003), on transcribing video data:

1. The first step required viewing and re-reviewing (reading and re-reading according to Smith et al 2012). Powell et al (2003) argues that this step requires viewing of video data attentively to become familiar with the contents of the video.
2. "Initial noting" (Smith et al 2012:83): In this phase, video data is describing in such a way that anyone reading the transcript will gain a clear idea of what happened. Data was just descriptive not interpretative or inferential (Powell et al 2003).
3. The third stage involves development or identification of emerging themes within the context of my research questions. The original whole of the transcript becomes a set of parts as one begins to conduct the analysis (Smith et al 2012).
4. After identifying themes, stage 4 involves the development of a charting or mapping of how the themes fit together. The analyst searches for connections across emergent themes (Smith et al 2012). Some of the themes may be discarded at this stage depending upon the overall research questions and their scope.
5. Moving to the next case. After transcribing the first video clip, stage 5 involves moving to the other clips and carrying out stages 1 to 4 again. Though each case is treated in its own terms, it is possible that the transcription of these videos may be influenced by themes identified in the first script (Smith et al 2012).

6. Looking for patterns across cases. This stage may involve laying each table or figure out on a large surface and looking across them for any similarities. One looks for connections across cases illuminating the most potent themes (Smith et al 2012).

The next section describes how data was transcribed and analysed in this study following the six stages articulated above.

3.5.1.1 Viewing screencast video data attentively

To become familiar with the content of the screencast video and audio data, I watched and listened to the clips several times. I repeatedly viewed screencast video and audio clips and compared it with field notes and notes from interviews to enhance triangulation in my data analysis. The first phase, involved watching and listening without intentionally imposing a specific analytical lens on my viewing (Powell et al 2003). The goal was to become familiar with the recorded sessions in full and ensure that the participant became the focus of the analysis (Smith et al 2012).

3.5.1.2 Initial noting

The use of screencast video data from JING resulted in an enormous amount of information, which posed a challenge of knowing it in fine detail. In this phase of describing data, the first written transcripts were just descriptive and not interpretive or inferential. The focus was on producing a comprehensive and detailed set of notes. It was a straight case of stating what was seen on the screen and what discussions were recorded. In some instances, I reinterpreted and modified participants' comments to provide a smooth flow of the narratives. Powell et al (2003) and Smith et al. (2012) say that the idea of this phase is to describe the data so that someone reading the descriptions would have an objective idea of the contents of the screencast video clips. Attentively watching and listening to the screencast videos helped me to become more familiar with the stream of participant discussions.

3.5.1.3 Developing emergent themes:

The data analysis phase consisted of carefully reviewing the screencast video clips and identifying emerging themes within the context of the research questions. An aspect

was considered a theme if it either confirmed or disaffirmed research questions (Kiczek, 2000; Maher, 2002; Maher & Martino 1996a) or was significant in its relation to particular research questions being pursued (Glaser & Strauss 1967).

Sequences of emerging themes were analysed, thus building narratives and interpretations of what was observed in screencast video clips. The list of themes that were considered draws from Ruthven et al. (2009) emerging themes on the contribution of technology to the learning process of mathematics. These themes include: (i) increased speed and efficiency; (ii) supporting processes of checking, re-trialling and refinement; (iii) enhancing participants' sense of capability; (iv) facilitating clear organisation and vivid presentation of materials (v) enhancing the variety and appeal of classroom activities; and (vi) fostering individual independence and peer exchange. I used these themes as a starting point for my data analysis.

Themes (i) and (ii) related to the first research question: How do participants employ GeoGebra to support their understanding of modulus functions and their properties? Themes (iii) – (vi) tried to address the second research question: How does GeoGebra relate to and contribute to the whole learning experience?

I also jotted down some instances, outside the research questions that emerged as participants were investigating modulus functions.

3.5.1.4 Connections across emergent themes:

Atkinson and Heritage (1984) and Erickson (1992) say that transcription of videos can be selective but theoretically guided by the research questions. Transcription is a process of transferring data from video recordings to a page. I transcribed screencast video clips to identify connections across participants' understanding of modulus functions. In transcribing screencast video clips, I tried not to give my own perspective of the events, but focused on those aspects that I considered relevant to the research questions. The transcripts produced were a close approximation of interactions and discussions that took place during the sessions. I kept transcripts in appropriate narrative reports, to provide evidence of findings in the participants' own words. During this stage, I transcribed events, to analyse elements of discussions and flow of ideas as

well as screencast video presentations. I also analysed portions of screencast video clips data to provide evidence for important answers to the research questions.

3.5.1.5. Moving to next cases:

I transcribed more screencast video clips from other pairs, noting anything not observed from the first clips and bracketing ideas already identified.

3.5.1.6 Looking for patterns across themes:

Looking for patterns helped me to interpret data. This activity is similar to identifying emerging themes, which requires watching screencast video clips intensively and closely for long period. During this phase of the analysis, I focused my attention on looking for patterns across themes to highlight the contributions of GeoGebra in the teaching of modulus functions. The developed patterns focused on my research questions.

I annotated the transcripts with codes (T1, T2 etc.).

T1 focused on the speed and efficiency with which participants tackled tasks in GeoGebra. It included:

- Number of tasks tackled with GeoGebra within a given space of time;
- Methods of data input:
 - Input screen;
 - Tools on the menu.
 - Use of mouse
 - Use of sliders
 - Dragging

T2: This included any aspects that supported processes of checking, re-trialling and refinement. These included:

- Instances in their work where participants used GeoGebra and for what purposes (e.g.)
 - To deduce common definitions of modulus functions.
 - Out of curiosity to check what graphs of modulus functions looked like.
 - To verify graphical representations or solutions to equations and/or inequalities;

- Sharing statements or comments unrelated to the discussion topic.

T3: Any instances which highlighted and increased participants' capabilities to tackle a problem with the aid of GeoGebra. These included:

- The use of GeoGebra to tackle more complicated problems;
- The use GeoGebra to extend their competencies on modulus functions?

T4: Focused on clarity in organisation of work and the use of multiple representations when solving problems;

T5: This focused on instances where the variety and appeal of classroom activities was observed;

T6: These are instances of individual independence and peer exchanges while using GeoGebra.

The identified patterns were defined in relation to the research questions pursued in this study.

3.6 TRUSTWORTHINESS OF THE STUDY:

There is no accepted consensus about the standards by which qualitative research should be measured. The quality of a study, from a qualitative perspective is concerned with the soundness or trustworthiness of findings of the study, warranted by the methods that have been employed. Since the research concepts of reliability and validity are rooted in positivist perspective, Guba and Lincoln (1985) substituted them with 'trustworthiness'. According to Guba and Lincoln (1985), trustworthiness contains four aspects: credibility, transferability, dependability and confirmability. These four aspects were adopted for this study.

Credibility: The mathematics topic selected for this study (modulus functions) was from the participants' curriculum. All research activities fitted well within the college schemes of work. The sample selected best represented the population. I used multiple data collection methods (i.e. on screen recordings, observations and interviews) synonymous with qualitative research methodology, in order to enhance credibility in the findings. Field notes were collected during lesson observations and most lesson activities were video recorded. I adhered to an open-ended perspective of constructivism, by allowing participants to assist with data collection (Johnson 1997). Participants were shown how

to record on-screen activities and their conversations. Video clips used in this research, were recorded by participants. I complimented field notes with video recordings and cross checked my interpretations of video data through group interviews. Johnson (1997:284) argues, “engaging multiple methods such as observations, interviews and video/audio recordings leads to a more valid, reliable and diverse construction of realities”. Participants were allowed to verify transcripts of the recordings. Research findings were taken back to the participants to verify the accuracy.

Transferability: The burden of transferability lies with the reader (Mertens, 2010) however, sufficient detail on how data was collected and analysed is discussed in this chapter. To allow transferability, I provided sufficient detail of the context of the activities carried out during this study, to enable readers to decide whether these activities are similar to situations in which they are working and whether they can be replicated.

Dependability: The collection of data and continuous processing and analysis of data during the course of the research allowed for verification of some themes that emerged from the recordings.

Confirmability: The topic selected for the study was from the ‘A’ level mathematics curriculum (see Appendix H). I took steps to demonstrate that the findings emerged from data collected from participants’ experiences, not from my own predispositions. Where ever possible, data was presented in narrative vignettes before being analysed. Findings that emerged from the research can be reconfirmed by replicating the same activities with different participants using the same technology.

Trustworthiness in qualitative research can be facilitated through the construction of appropriate methods of data collection and analysis (Sarandakos 2005).

3.7 MY ROLE AS RESEARCHER:

During sessions I assumed the role of a participant observer and facilitator even though I retained my role as the lecturer in charge of the sessions. According to constructivism, teachers in a classroom should be facilitators who help learners understand mathematics concepts or solve problems independently (Gamoran, Secda & Marrett 1998). They define facilitators as supporters, providers of guidelines, people who ask

questions and talk with learners. In this study, I set the goals for the sessions, explained briefly the tasks to be carried out and participated in discussions when invited by the participants.

Cohen et al (2003) recommend the use of participant observations in studying small groups for a short time to derive data from 'real life' (Robson 2009:189) settings. Monaghan (2004) recommends researchers to study social interaction through participant observations. I was a participant observer, in the sense that participants carried on, with their activities, with very little assistance from me. In cases where I joined in the conversations, it was to follow up a point picked up from a pair's discussions. Adopting a participant observer role was particularly helpful in my data collection. Close contact and immersion in some discussions was necessary for understanding the meanings and actions of participants during the investigations and the processes by which participants constructed conjectures from their investigations. Through participant observation, I had a chance to observe participants working with GeoGebra and check the technical aspects of their recordings. Participants worked in a GeoGebra environment without seeing me as an intruder (Robson 2009). However, participants viewed me as an authority, judging by the number of times they consulted me for clarification on some aspects of modulus functions. In most cases I avoided giving direct answers but referred the question back to the participants, prompting them to use GeoGebra for further investigations.

3.8 ETHICAL ISSUES:

According to Kumar (2005:192) "ethical" means principles of conduct that are considered correct, especially those of a given profession or group. The principles of conduct are the most important as they address the issue of content of ethical behaviour in a profession. In conducting this study, I followed the ethical guidelines laid down by the British Educational Research Association (BERA 2004).

I carried out this study as a practitioner researcher working with a small group of six "A" Level participants in the College where I was based. Having initiated a project that was relying on participants, the importance of ethical conduct could not have been taken for granted. I took precautions to ensure that the balance between my professional and

research role, as an insider was maintained. Being an insider researcher, I was familiar with the College setup and had established relationships with participants from their previous year of study. I also had access to their previous achievement records and their individual profiles. It was imperative to observe standard ethics such as obtaining permissions, maintaining confidentiality and guaranteeing concealment of identities (Denscombe 1998 in Bell 2006).

Before the study began, I communicated the purpose of my research to the participants and the College authorities, making it clear what the study was all about and what it was that I wanted to do with the participants. I made it clear to the participants that neither their academic needs, nor my assessment of their academic work would be compromised by their participation in this study. The Head of the college and the Curriculum Leader (Mathematics) were assured that GeoGebra software would only be used when it fitted with the C3 – Core mathematics learning objectives on modulus functions. I also took into consideration recommendations from the Ofsted (2012) report that participation in activities involving the use of technology should not be for the sake of using technology but should add value to participants' learning, foster problem solving techniques and promote high order thinking skills.

To ensure that participants made an informed decision, I provided them with the framework and aspects of the study, describing objectives and the intended outcomes for each session (See Appendix G). I also made it explicitly clear to participants that they were not under any pressure to participate in the research. However participating in each session, would be assumed as consent granted. I sought informed consent from three participants who were 18 and over at the time of the research and parental consent for three of the participants below the age of 18. Diener & Crandall cited in Cohen (2003:51) define informed consent as 'that procedure in which individuals choose whether to participate in an investigation after being informed of facts that would be likely to influence their decision'. Seeking informed consent meant that I made sure that participants were fully informed and understood what it meant to have their onscreen activities and conversations recorded, and that they had consented to the intended uses of the recorded data.

Capturing information pertaining to participants' classroom activities with videos in this research introduced more ethical issues, including informed consent before the start and during the process of recording classroom activities. Roschelle (2000) suggests obtaining progressive levels of consent, as and when needed. I sought consent before recordings commenced. However, participants were informed that any unanticipated recordings acquired accidentally, such as some off-the-activity conversations or use of bad language would not be used in the research (Roschelle 2000).

Participants were accorded the right to access a complete rendition of the video transcripts upon which the analysis was based, for them to verify if this was a true interpretation of their work.

I also sought informed consent from the Head of College and the participants' parents to record lessons activities during the sessions. Participants were fully informed through the consent letter about the implications of having their mathematical activities captured on screencast videos and the intended use of the recordings.

Within my capacity as a participant researcher, I only guaranteed anonymity where it was possible but was honest when this guarantee was not possible to maintain. This being a study involving a few individual participants, I only guaranteed anonymity and confidentiality through use of pseudonyms (Cohen 2003) in the final report. I used all screencast video and audio recordings made in the research for writing the report only. No tapes were to appear in the final report. I kept all tapes securely in my home for a period of six months after the report, in case I wanted to cross-reference some information. Thereafter all recordings were to be destroyed. Participants reserved the right to read transcripts from the JING recordings, to see if they were a true reflection of what had transpired during lessons.

I informed the Head of Academy that anonymity, in some instances could not be guaranteed even if a pseudonym was used for the college. In a case where the findings of this study were published, the report will be published under my real name and becomes part of a research work in the public domain. The details will become hardly

confidential, since anybody who knows me and knows where I teach would also know the identity of the school.

3.9 SUMMARY:

This chapter provided a theoretical approach for the methodology adopted for this research. It outlined sampling techniques, details of participants' profiles, methods of data collection and analysis. Each method of data collection used in this study was discussed. The chapter also outlined how screencast video data collected using JING, data collected from field notes and data collected through interviews was transcribed and analysed following an interpretive phenomenological analysis framework.

CHAPTER 4:

4.0 DATA COLLECTION AND DATA ANALYSIS:

In this chapter, I consider data drawn from a sequence of six sessions in which participants worked mostly in pairs and investigated a series of modulus function problems using GeoGebra. The chapter outlines, describes and comments on the activities carried out during the six sessions of the study. In each session, I highlight how participants used technology (in this case, GeoGebra) to learn different aspects of modulus functions. My data consists of extracts from screencast video clips captured during some sessions, field notes taken during observations, group interviews and discussions held throughout the sessions and some scripts from participants consisting of pencil and paper sketches and hand written scripts from set exercises. The sessions are in chronological order but some of the activities within the sessions are not. I tried to arrange them in a way that highlights the learning process as participants went through the investigations.

My sample consisted of six 'A' Level (A2) participants purposefully selected for this study. We used GeoGebra to examine participants' experiences of using technology in a learning process. Participants worked in pairs and were encouraged to discuss aloud for screencast video and audio recording purposes. Such an instruction made participants conscious that they were being recorded, hence discussions were, at times, controlled and focused on the task during the first few activities.

Each of the six sessions was divided into sequential activities. The average length of each activity varied, depending on the complexity of the problem under investigation. The sequence of activities was, at times disrupted when unexpected graphical representations of modulus functions appeared on the screen and participants became curious to investigate further along the lines of the emerging themes.

Working in pairs became less and less overtime, as the six participants became more familiar with the work on modulus functions and preferred to work individually. Four of the participants managed to download the GeoGebra App on their Android and Smartphones, which reduced the use of laptops and pair-work. In such instances, I

relied on field notes taken during the sessions. I maintained the role of a participant observer during all the six sessions of the study.

Each session comprised of activities and each activity had given guidelines or set questions. However, at times participants went beyond set tasks and exercises.

4.1 SESSION 1: Introduction to GeoGebra and the modulus function:

In a classroom setting, a mediated mathematical action is grounded in the way various tools such as mathematical symbols, technology and language, are appropriated (Rogoff 1990). Session 1 was divided into three activities. The first activity dealt with familiarisation with GeoGebra but focusing on how to appropriate various tools that were relevant to the concept of modulus functions. This activity focused on introducing GeoGebra as a tool in the participants' learning environment. The second activity focused on using GeoGebra to investigate graphs of the functions $y = |ax + b| + c$. Activity 3 was used as an enhancement and checking activity to check participants' level of understanding of modulus functions, based on the work covered in the first two activities.

The lesson started with an arrangement of the six participants in pairs. An Interactive Whiteboard was used to introduce GeoGebra tools relevant to the learning of modulus functions and to discuss the layout of the multiple screens of GeoGebra. A few simple algebraic functions were used to demonstrate the screen layout. Instead of giving direct explanations on how to sketch the graphs involving modulus functions, participants were guided through questions that focused on the development of the concept of modulus functions (see Appendix A). The questions were selected from an exercise in Neill and Quadling (2003:282) textbook, to allow participants to familiarise themselves with the GeoGebra software while at the same time exploring the concept of modulus functions.

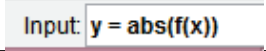

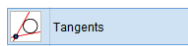
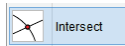

At this stage, participants were introduced to JING, a screencast video capturing software that was used to record some on-screen activities and discussions from participants. Participants were allotted a short period to playfully explore the recording software. The first ten minutes generated some chuckles and amusements as

participants recorded and played back their recordings. During the ten minutes, participants' attention was more focused on the recording software than GeoGebra.

4.1.1 Activity 1:

In this first activity, we used laptops, Interactive Whiteboard and an overhead projector. The activity was devoted to learning how to access input registers and sketch modulus functions in GeoGebra. Participants had encountered GeoGebra (the software) while doing a piece of Core 3 coursework on 'Numerical methods', therefore they were familiar with certain features of the software. The focus of this activity was to introduce participants to tools used in the process of learning modulus functions. The activity was initially concerned with the basic GeoGebra tools listed in Table 4.1, i.e. algebra input; movement tools like sliders keyboard arrows; point tool; tangent tool, how to get coordinates of points of intersection; and how to change scales on the axes using the mouse. I also demonstrated tools for personalizing presentations e.g. how to change colour or line width.

Table 4.1: List of some of the artefacts and input language introduced.

Description	Syntax or symbol
Input window	
$y = 2x + 3 $	$y = \text{abs}(2x+3)$ or $y = \text{abs}(2*x+3)$
$y = x^2 $	$y = \text{abs}(x^2)$
Keyboard Arrows	
Differentiating $f(x)$	$f'(x)$ or differential($f(x)$)
The tangent icon	
intersection icon	
Slider icon	

Before formally introducing tools listed in Table 4.1, participants received instructions to:

- Type-in a modulus function in the Input View
- Observe the graphs in the graphic view, and their related algebraic representations in the Algebraic View.
- Make comments on structure of the graphs, and

- Test if comments were valid by inputting more functions.
- Explain observations, highlighting situations in which initial comments were either true or false.

I selected a few screencast video clips from the recordings that provided rich data that highlighted participants' experiences of using GeoGebra. The screencast video clips were transcribed and presented in short narratives (Vignettes).

Vignette 1 outlines extracts of transcripts from a screencast video clip captured within the first few moments of the session, when Peter and Susan started entering functions on a GeoGebra screen before the formal introductions of the tools.

Vignette 1: *Becoming familiar with GeoGebra's inputs (Instrumentation process):*

Susan and Peter (Peter is in charge of the keyboard)

Line 13: *$y = x$ is typed in the Input window.*

Line 14: *A straight line appears in the graphics window while an algebraic representation $a : y = x$ appears in the algebra window;*

Line 15: *$y = |x|$ is typed. ["Invalid input" appears on the screen];*

Line 16: Peter: "Ooohs. What have I done wrong?"

Line 17: Peter: "Let me use $\text{abs}(x)$ instead of the vertical bars. That's what it is called on the graphics calculator";

Line 18: *$y = \text{abs}(x)$ is typed and a graph appears in the graphics window and the equation $f(x) = |x|$ appears in the algebra window (See Fig 4.1);*

Line 19: "Great!" Peter shouts out to the rest of the participants! "Type in $y = a... b... s...(x)$ in place of vertical bars". [Peter spells out the word]

Line 20: Me: "Don't forget to write down the description of the graphs you see on the screen." [I prompt participants].

Line 21: Susan: "The graph of modulus function is V shaped. This one has a vertex at $(0, 0)$,"

Line 22: Susan: " Hmmm. Let's try another function";

Line 23: [Clicks from keyboard]. *$y = \text{abs}(x + 3)$ is typed in the input window;*

Line 24: Another V shaped graph appears in the Graphic window and $f(x) = |3x + 1|$ appears in the Algebra window.

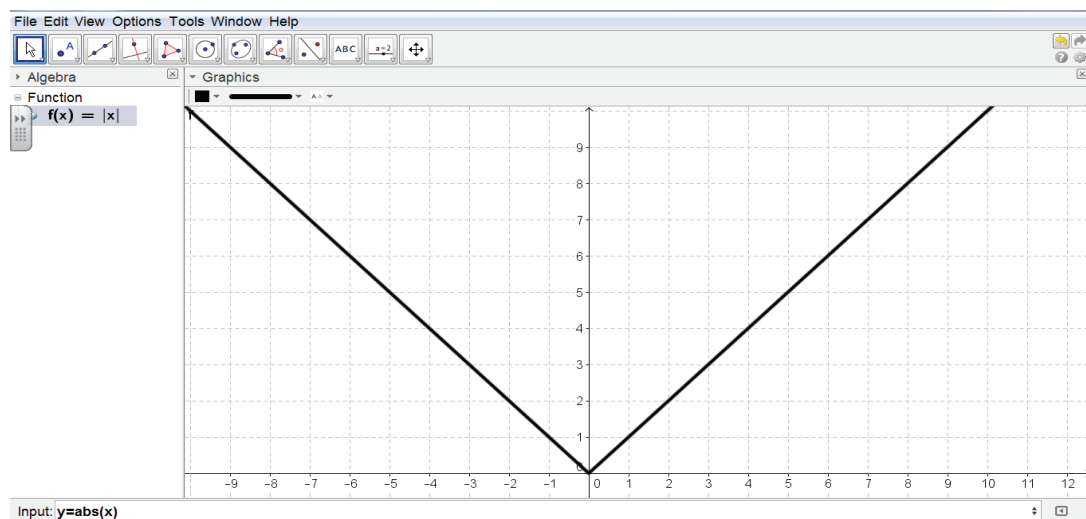
Line 25: $y = (2x - 4)$ is typed,

Line 26: Another V shaped graph appears and a function $g(x) = |2x - 4|$ is displayed in the algebra window (Fig 4.2),

Line 27: “All the graphs of modulus functions are V shaped” [Audio].

Each time $y = \text{abs} [<\text{Function}>]$, is typed in the Input window, GeoGebra automatically assigns it to different labels: $f(x)$ or $g(x)$ (see Fig 4.2). This saved participants from searching for notation to use each time they entered different functions on the same screen.

Fig 4.1: Graph of $f(x) = |x|$



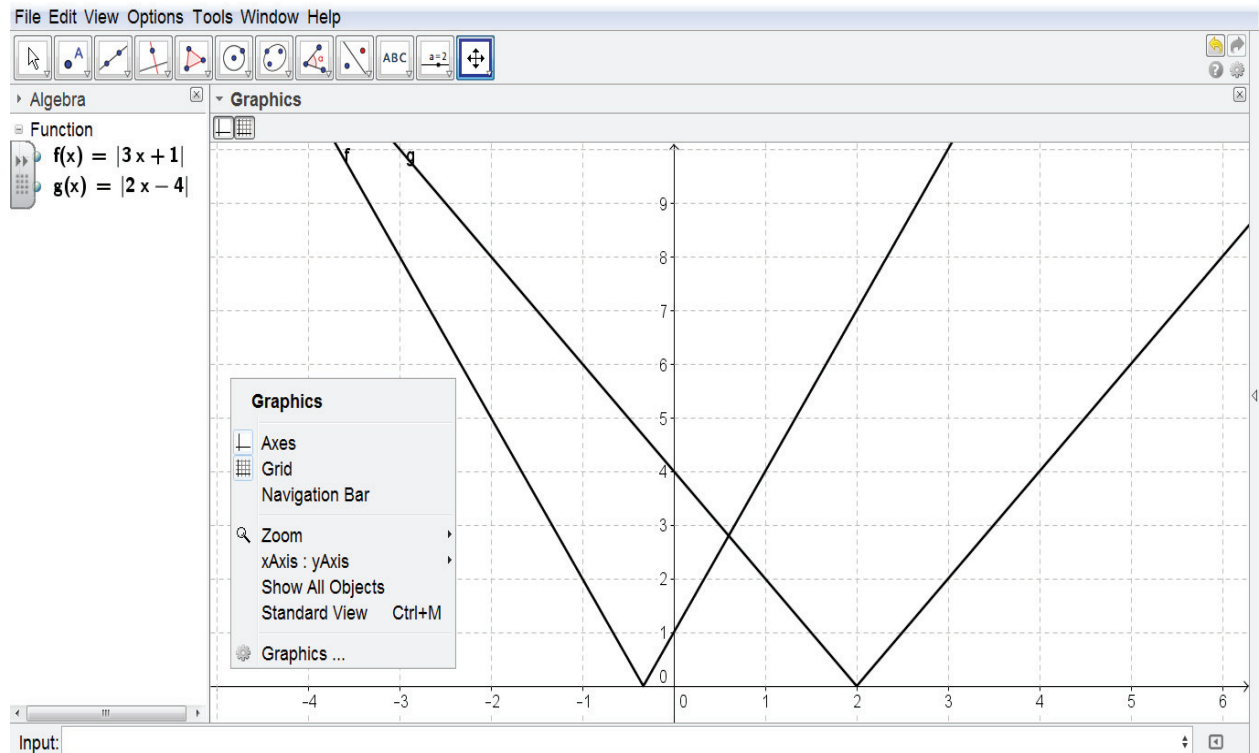
The vignette highlights participants' process of becoming familiar with the input aspects of GeoGebra. There were minor delays in finding the correct syntax for modulus function at the beginning as highlighted in the comments in Vignette 1. Participants identified correct syntax for inputting modulus functions without my assistance and continued with more investigations.

At this early stage of the investigation, participants were confronted with three different semiotic registers. GeoGebra requires some specific format of input, as discussed in Vignette 1 (Lines 15 - 18). The Input register only accepts language “ $\text{abs}(f(x))$ ” (see

Line 25). The output registers, the algebra screen, converts the input $y = \text{abs}(2x - 4)$ and gives it as $g(x) = |2x - 4|$ (see Line 26), while the graphic screen displays the graphical representation of the function (see Fig 4.1). The algebraic expression and its corresponding graphical representation require interpretation by the participants. However, in this activity, participants focused on finding the correct syntax and there was no discussion about the presentations in the graphics screen or any question about why it is different from linear graphs. There are a few lines (see Line 21 and Line 27) in the conversation where there is mention about the shape of the graph, but no formal description is given.

Within the process of familiarising with the software, participants are already moving beyond the task by trying modulus functions other than $y = |x|$ (see Lines 23 – 26).

Fig 4.2: *Modulus function graphs of $f(x)$ and $g(x)$ as they appear on the screen.*



Vignette 1 illustrates participants' journey as they learnt how to use GeoGebra, while at the same time exploring the modulus function concept and getting a glimpse of the graphs of modulus functions. The same process was observed in Vignette 2. Peter and Susan established the link between the symbols " $| |$ ", the abbreviation "abs" and the term "modulus value". This highlighted instances where participants were getting familiar with GeoGebra, discovering correct syntax that enabled them to communicate with the software throughout the duration of the study.

As participants entered more functions on the same screen (Fig 4.2), it became difficult to distinguish, which function in the algebra window represented which graph in the graphics window. At this point, I demonstrated on the Interactive Whiteboard, a facility in GeoGebra, which allows colour coding of graphs and their respective functions.

Vignette 2 is a narrative of Peter and Susan's work as they personalised the graphs and investigated more functions, selected from a list of questions given in exercise 1 (See Appendix A).

Vignette 2: Personalising graphical representations on the GeoGebra screen:

Susan and Peter continued...

.....

Line 35: $y = \text{abs}(2x - 4)$ is typed;

Line 36: $g(x) = |2x - 4|$ appears in the algebra section and another V shaped graph appears in the Graphics window (See Fig 4.2);

Line 37: $y = \text{abs}(3x+1)$ is typed;

Line 38: $f(x) = |3x + 1|$ appears in one window and another V shaped graph appears in the graphics view.

Line 39: Susan: "Each time we type in y is equal to some function, the output comes up as $f(x)$, $g(x)$ or $h(x)$."

Line 40: Peter: "That's machine generated. We do not need to worry about naming the functions $f(x)$ or $h(x)$."

.....

Line 40: Mouse pointer moves onto one of the graphs and a little menu appears on the screen (see Fig 4.2).

Line 41: Mouse clicks on 'Object Properties' on the screen;

.....

Line 45: A few more clicks and the graph changes colour to red (Fig 4.3)

[The corresponding function in the algebraic window changes to red as well]

Line 46: Another graph turns blue (see Fig 4.3);

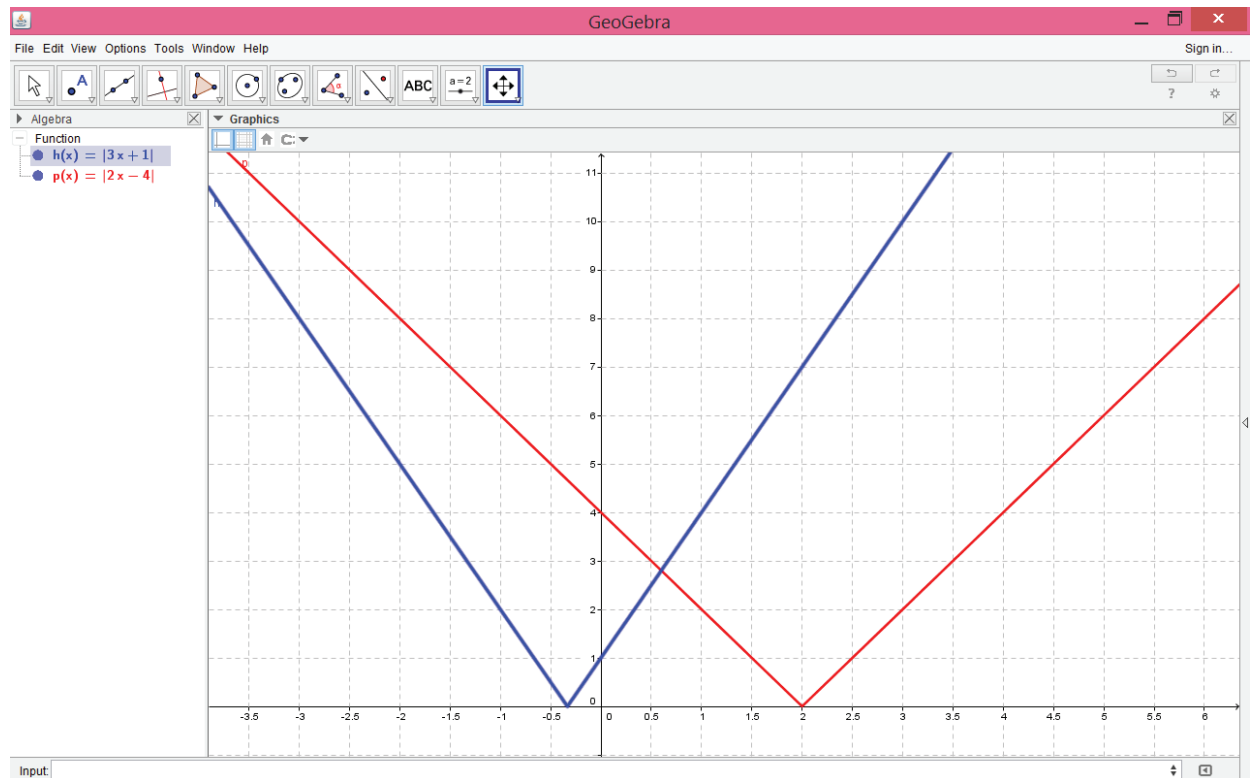
Line 47: Cursor moves around the screen, a graph turns red and another one turns blue.

Line 48: The cursor moves onto the horizontal axis and it is dragged to the right of the screen.

Line 49: As the cursor is dragged, the x-axis scale is stretched giving a 0.5 interval.

Within this vignette, participants learnt how to colour code representations and how to change the scales on the axes using a combination of the 'move' tool on the menu bar and the mouse. This enabled participants to enlarge or shrink the graphs for better view.

Fig 4.3: Graphs of the functions $q(x)$ and $h(x)$.



GeoGebra provided an opportunity to observe two different presentations of the same mathematical concept in the algebra and graphics windows. Manipulations carried out in one window, were immediately updated in another and the impact of the actions taken upon the graphical representation was observed on the corresponding depiction in the algebra window. Colour codes (see Fig 4.3) linked multiple representations in the two algebraic and graphics windows. Any dynamic changes happening in one window could be easily associated with changes in another window. When more functions were displayed on the same screen, colour codes presented easier-to-follow relationships between the multiple representations. Bayazit and Aksoy (2007) argue that representing the same concept in two different ways on the same screen promotes learners' depth of understanding and development of knowledge across the representations.

Activities recorded in Vignette 1 and 2 indicate that participants quickly adapted to the use of GeoGebra with minimal instructions from me. Though not discussed in depth, these excerpts of discussions show the emergence of the idea of the shape of the graph of $y = |f(x)|$, informally described as 'V' shaped.

Vignette 3 is a short narrative of participants trying to describe modulus function graphs.

Vignette 3: *Investigating graphs of modulus functions:*

Susan and Peter continued

.....

Line 52: *Peter: “Both graphs are V shaped”.*

Line 53: *Susan: “They all turn at the x-axis”.*

Line 54: *Me: “Focus on one function, for example, $y = \text{abs}(x + 3)$ and see how it relates to the graph of $y = x + 3$ ”.*

Line 55: *$y = x + 3$ is typed on the same screen,*

Line 56: *A click on ‘Object properties’ and a dotted line appears (see fig 4.4)*

Line 57: *Peter: “It seems like some points are reflected in the x – axis”.*

Line 58: *Susan: “Look! point A(-4, -1) goes to B(-4, 1); C(-5, -2) to D(-5, 2)”*

Line 58: *Peter: “Points A, C, E and G, are reflected in the x-axis to B, D, F and H respectively”.*

Line 59: *Susan: “It’s only those points of the graph of $y = x + 3$, below the x-axis that have been affected”.*

Fig 4.4: The graphs of $y = x + 3$ and $f(x) = |x + 3|$.

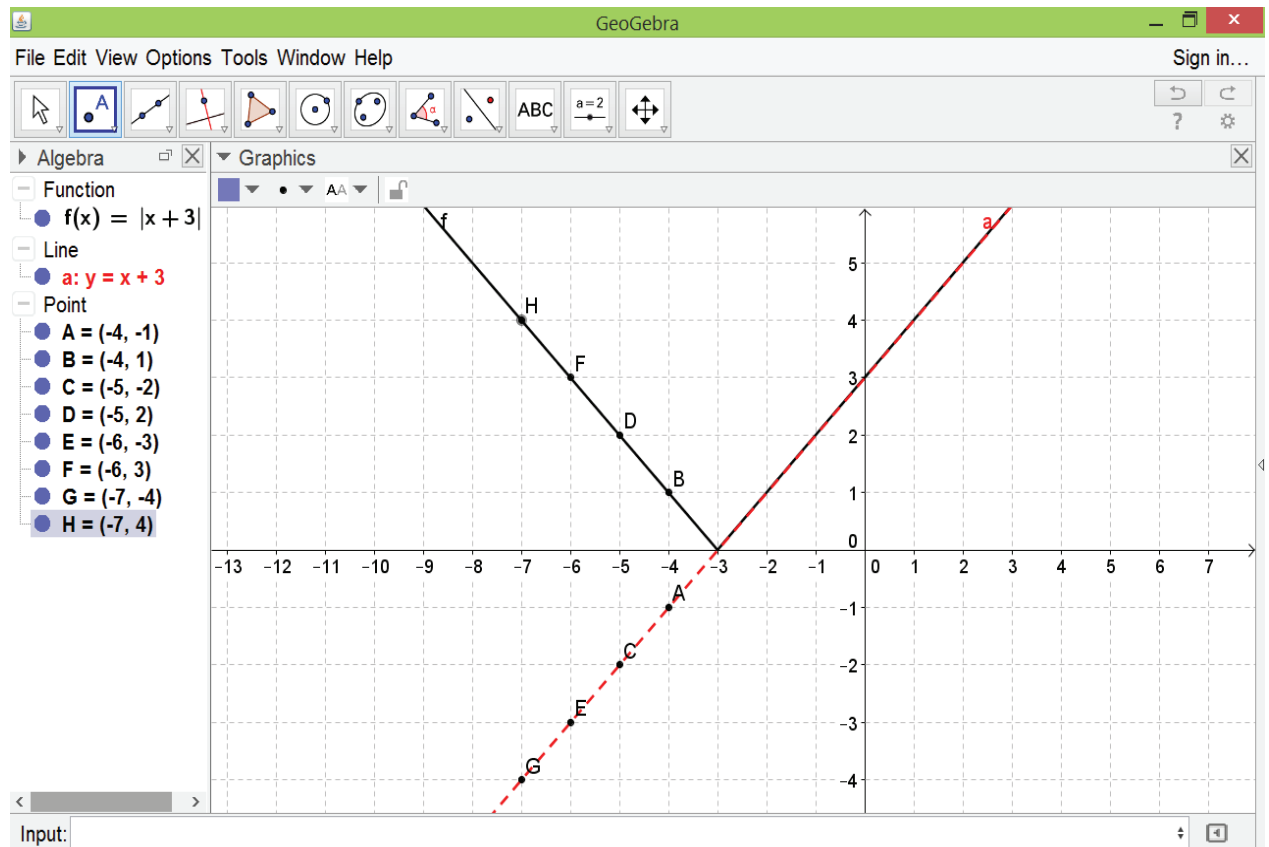


Fig 4.4 displays the graphs of $y = x + 3$ and $y = |x + 3|$ and clearly displays coordinates of each point A, B, C etc. in the algebraic window. GeoGebra displays clear and accurate graphical representations providing participants with an opportunity to come up with rough descriptions of graphs of modulus functions. It has afforded them opportunities for further investigation without consuming a lot of time. Participants entered more functions and viewed more graphical representations on the screens, from which they deduced common features related to properties of graphs of modulus functions (V shaped, turning at the x-axis and reflected in the x-axis).

From a summary of discussions noted on screencast video, participants made the following observations:

- the part of the graph of $y = x + 3$ for $x \geq -3$ remains the same as that of $f(x) = |x + 3|$,
- But the part of the graph of $y = x + 3$ for $x < -3$ is reflected in the x-axis.
- All values of y below the x-axis are reflected in the x-axis.

Peter and Susan used accurate visual representations in Fig 4.4 to verify their hunches that the graph of $y=|f(x)|$ turns at the x-axis and does not exist below the x-axis.

What is interesting to note is the point to point comparison of points apparently being done from use of accurate diagrams produced by GeoGebra, based on the idea of the dynamic relationship between points on the graph of $y = x + 3$ and $y = |x + 3|$. Analysis of the data from activities shows the role of GeoGebra in the early formulation of mental images of the graph of modulus functions. The display of graphical representations on a grid made it easy to compare coordinates accurately. This is evidence to support the view that GeoGebra facilitates the construction of mental images of graphs of modulus functions.

A few themes emerged from the analysis of conversations and on-screen activities in Vignettes 1, 2 and 3. GeoGebra failed to recognise $|x|$ as a valid input and participants had to find alternative coding to represent the same. GeoGebra recognised 'abs(x)' in place of $|x|$. An entry of $y = \text{abs}(x)$ in the Input window yielded $f(x) = |x|$ in the algebra window, which was totally different from the input. GeoGebra also recognised $f(x) = \text{abs}(x)$ as an input.

There was a process of meaning-making taking place in terms of input language and output language. GeoGebra displayed three different representations for the same function: $y = \text{abs}(x + 3)$ for Input; $f(x) = |x + 3|$ in the Algebraic window and the 'V' shaped graph in the graphics windows. Panasuk (2010) suggests that one of the indicators of conceptual understanding is the ability to recognise structurally the same relationship posed via multiple representations. In the absence of data from participants' contributions, it was difficult at this stage to recognise if participants understood the different representations displayed on the screen. In many instances, as shown on the screen, participants entered a function and moved on to the next function without discussing the link between the representations.

In Line 19, Peter shouted to the rest of the group to enter $y = \text{abs}(x)$. I noted this as an instance when GeoGebra fostered peer exchange by providing opportunities for participants to share knowledge and provide mutual support to each other. Participants

also used GeoGebra to support processes of checking; refining and testing initial observations (see Line 23 – Line 26 and Lines 35 – 50).

Vignette 4 is a narrative from a clip that came towards the end of the activity 1. In this activity, John and Emma were investigating trigonometry function $y = \sin(x)$ involving modulus functions selected from the ten questions in Appendix A. I selected this clips because of its departure from the usual investigation of linear functions involving modulus.

Vignette 4: *Investigating trigonometry functions involving the modulus function:*

John and Emma

.....

Line 84: *John types $y = \sin(x)$;*

Line 85: *A click on the screen pulls down a 'Graphics' menu;*

Line 86: *Another Click on 'Graphics' and a few more clicks allow John to change the x-axis scale from rational numbers to π (Fig 4.5);*

Line 87: *$y = \text{abs}(\sin(x))$ is typed on the same screen and the graph of $f(x) = |\sin(x)|$ appears of the screen (Fig 4.6);*

Line 95: *A few clicks on the graph of $y = \sin(x)$;*

Line 96: *Pulls down a 'Function f' menu (see Fig 4.6);*

Line 97: *Selects 'Object properties' and formats the graphs;*

Line 98: *Parts of the graph of $f(x) = \sin(x)$ below the x-axis as a dotted line (see fig 4.6)*

Fig 4.5: The graph of $f(x) = \sin(x)$.

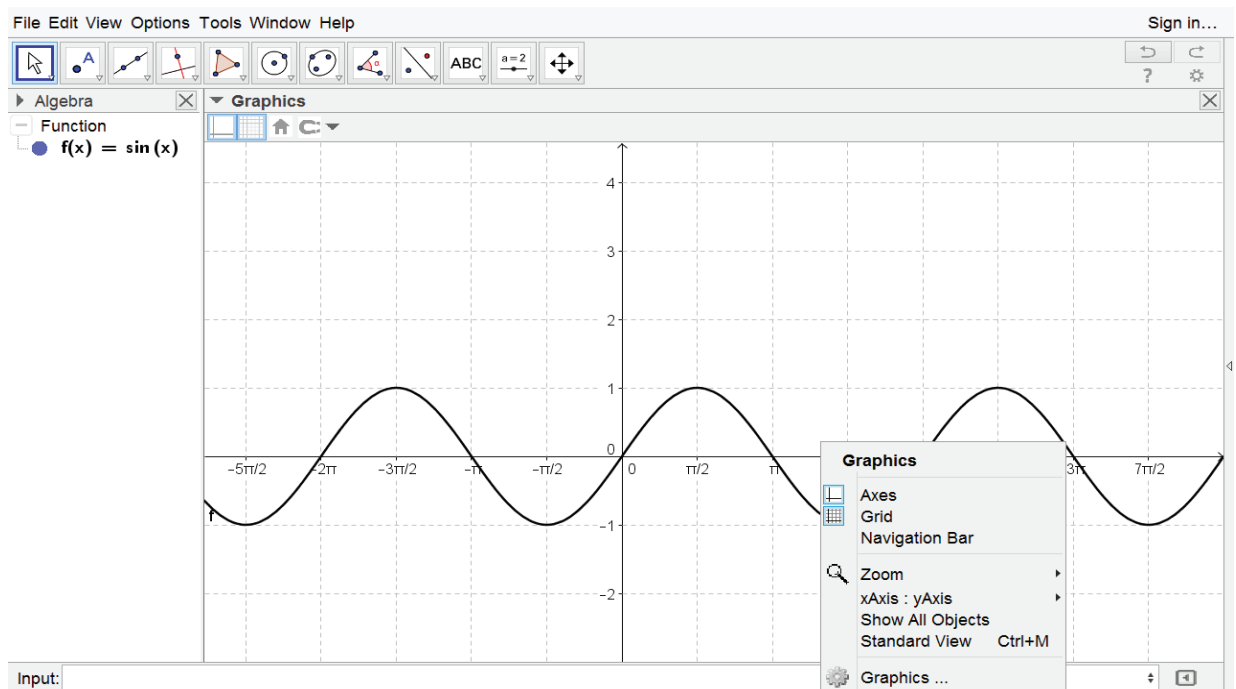
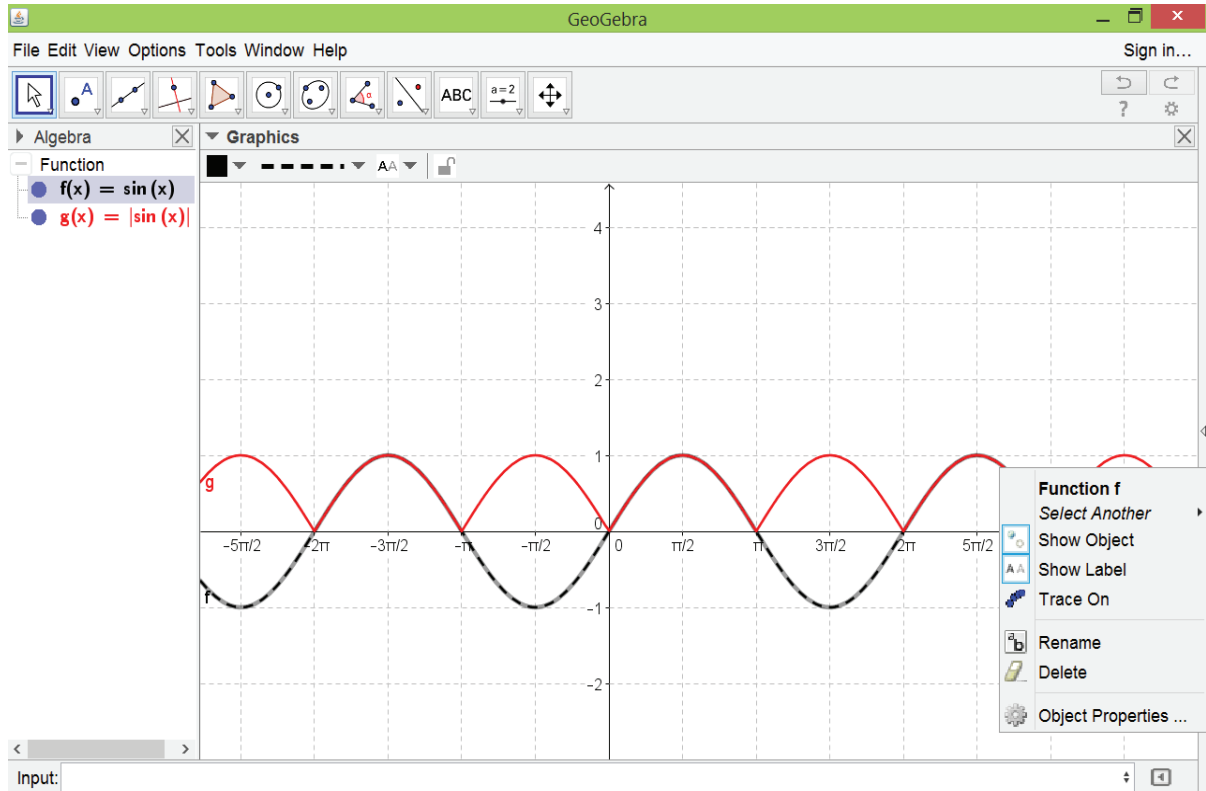


Fig 4.6: The graph of $f(x) = \sin(x)$ and $g(x) = |\sin(x)|$ on the same screen.



Traditional approaches to teaching modulus functions normally consider linear functions that involve modulus signs only. An analysis of participants' activities in Vignette 4 shows participants attempting graphs of modulus functions involving more complicated functions. The graph of $y = \sin(x)$ is a slight departure from the linear functions done by participants at the start of the activity. Displaying the graphs of $f(x) = \sin(x)$ and $g(x) = |\sin(x)|$ on the same window further confirmed the conjecture made by Susan and Peter that all parts of the graph of $y = f(x)$ below the x-axis are reflected in the x-axis. The graph of $g(x) = |\sin(x)|$ is different from the algebraic functions involving modulus functions that have been highlighted in Fig 4.1 to Fig 4.4 in that it does not have a distinct V shape.

Vignette 5: Refining participants' understanding of the properties modulus functions:

John and Emma continued

.....

Line 96: Emma: "The parts of the graph of $y = \sin(x)$ are reflected in the x-axis to give the graph of y is equal to modulus function of $\sin(x)$."

Line 97: Emma observes that the domain of the functions $f(x)$ and $g(x)$ remain the same but the range of $f(x)$ is $-1 \leq f(x) \leq 1$ while that of $g(x)$ changes to $0 \leq f(x) \leq 1$.

.....

Line 99: John: "Hhhmmmm!!!"

Line 100: John: "In other words, we can say that the graphs of modulus functions do not exist below the axis x-axis"

Line 101: Emma: "We have to be careful with that assumption. The graph of $y = |x| - 2$ was shifted 2 units below the x-axis [reference to earlier work in the activity].

Line 102: John: "For $y = |f(x + a)| + b$, the graph of $y = |f(x)|$ is translated 'b' units up or down and 'a' units to the left".

Line 103: Emma: "Yes but it's no longer the graph of $y = |f(x)|$. It's a translation of $y = |f(x)|$ and that gives you a function different from just the modulus function".

[Emma summons previous knowledge on transformations of function graphs]

Line 104: Emma: "... and my point is that graphs of modulus functions can exist below the x-axis. We cannot generalise that graphs of modulus functions do not exist below the x-axis".

Line 105: John: *“Okay then. We can say for all functions $y = |f(x)|$, the graph does not exist below the x-axis. ... but it will be different for $y = |f(x)| + a$, which involves some vertical translation of the graph”.*

Line 106: *“Let’s type in more functions to check if that is generally true”.*

At this point John and Emma entered more functions in GeoGebra to verify their observations. Through discussions, Emma and John began to formulate language (see Line 105) that provided a formal description of the graphs of modulus functions.

Participants still observed some sharp corners on the graph of $y = |\sin(x)|$. The phrase ‘V shaped’, previously used to describe the turning points was no longer used in this Vignette. They used the term “reflection” and also identified the line of reflection as the x-axis. Elements of a formal definition of graphs of modulus function were beginning to emerge. Participants however noted that there was one characteristic of the graphs of modulus functions that had remained consistent: The graph of the modulus function was above the x-axis.

There are two aspects of the sub-research questions that are highlighted in this activity:

- a) GeoGebra enhances participants’ sense of capability and confidence to tackle tasks beyond the ordinary algebraic functions (see Vignette 5). John and Emma extended modulus functions beyond what curriculum dictated. They displayed confidence by picking questions randomly from the worksheet and investigating their graphs using GeoGebra.
- b) GeoGebra was fostering peer exchange (see Line 96 – Line 105), notably by providing opportunities for participants to share knowledge as is further illustrated in the next vignette, where John and Emma also selected a question from Appendix A.

Vignette 6: Further investigation of the nature of graphs of modulus functions:

John and Emma continue

.....

Line 154: $y = -2\text{abs}(2x - 1)$ is entered;

Line 155: Fig 4.7 appears on the screen;

Line 156: Emma: "This V is upside-down and it's all below the x-axis now";

Line 157: John: "It's the opposite of the other graphs".

Line 158: Emma: "It's the effect of multiplying modulus function $f(x)$ by a negative number. That reflects it in the x-axis";

Line 159: Peter: "It's the graph of $y = 2|f(x)|$ that has been reflected in the x-axis".

Line 160: Peter: "What remains true is that it turns the moment it hits the x-axis".

.....

John and Emma concluded that the graph of $f(x) = -a|x - b| + c$ is an upside down V with vertex (b, c) , and gradient negative a for $x > b$ and gradient a for $x < b$.

Fig 4.7: Graph of $f(x) = -2|2x - 1|$

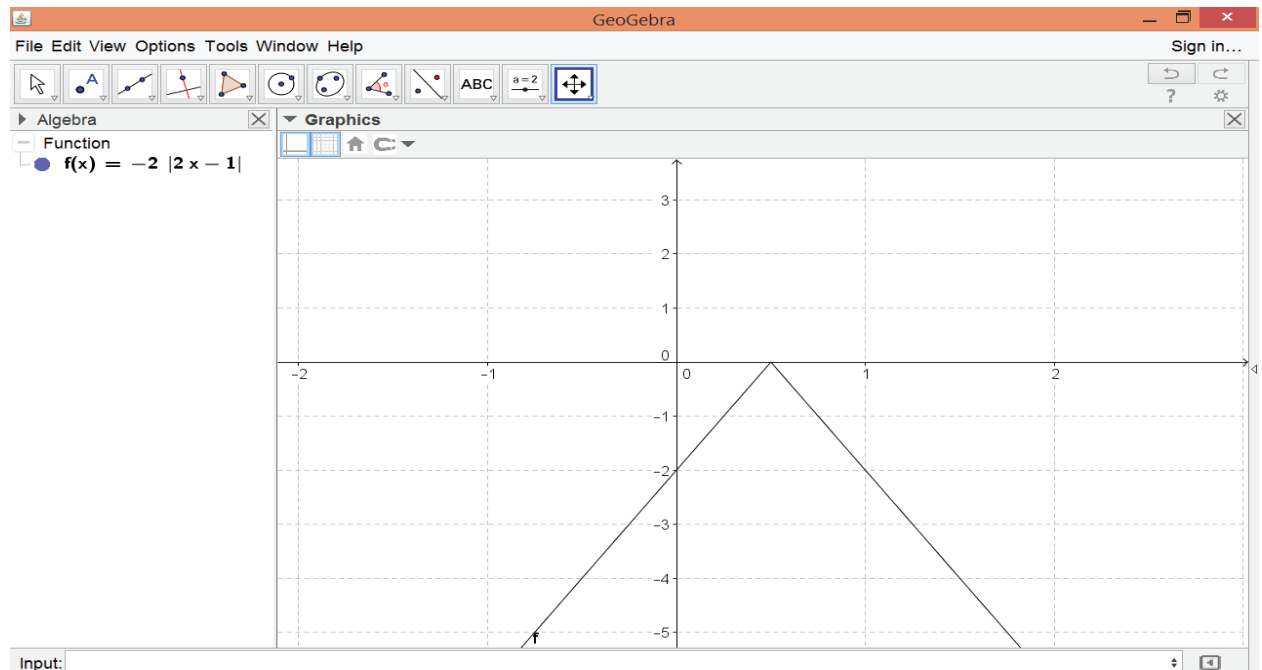


Fig 4.7 displays the graph of $f(x) = -2|2x - 1|$ from John and Emma's screencast video clip. It clearly showed the participants that there are some graphs involving modulus functions that exist below the x-axis. The representation dispelled the misconception that graphs of modulus functions do not exist below the x-axis as earlier highlighted (see Line 100: Vignette 5). John was of the opinion that if a function included an absolute value sign, then the graph must be above the x-axis. The same misconception was observed in a study carried out among 20 Turkish 16 – 18 years old students, (Monaghan & Ozmantar 2006), which noted a misconception that graphs involving modulus functions only exist above the x-axis.

Emma used GeoGebra to justify her argument and show John why she was right. She failed to come up with a convincing verbal argument in Vignette 5 but used technology to demonstrate her point. She takes advantage of the availability of technology to enter more functions of the form $y = a|x + b| + c$. Emma used GeoGebra (see Fig 4.7) to demonstrate that the graph of $y = -2|2x - 1|$ exists below the x-axis.

The first activity was meant for participants to familiarise themselves with GeoGebra. However, because of its practical nature and the use of modulus function questions, which participants had never encountered before, participants quickly moved on to explore the nature of graphs of simple modulus functions listed in Appendix A. Participants started making observations about the properties of modulus function graphs once they learnt how to input functions on the software. Once participants became familiar with tools and the input process, they went straight on to input modulus functions and started making comments about their observations.

There was also the element of increased speed of execution and improved accuracy of the presentations. GeoGebra, unlike pencil and paper, presented accurate diagrams and not sketches. Participants did not spend much time arguing about the graphical correctness of representations on the screens. GeoGebra gave participants opportunities to execute problems that were more complicated without even considering their complexity. Pape and Tochanov (2001) cited in Panasuk (2010) observed that when students generate representations of a concept while solving a problem, they

naturally tend to reduce the level of abstraction to a level that is compatible with their existing cognitive structure.

The amount of work covered in this activity was more than what was initially planned. Participants familiarised themselves with the features of GeoGebra but at the same time used the software to investigate less familiar modulus functions selected from Appendix A.

An analysis of video transcripts revealed that the participants engaged in activities, which had many features of exploratory work. They asked each other for information and opinions; they sought definitions and patterns and discussed ideas about modulus functions. Participants shared their thoughts and verified their observations through inputting more functions in GeoGebra. Any challenges met were generally constructive and all participants were involved in working towards making sense of the modulus functions, at first based on the graphical representations, then relating it to algebraic representations to generalise.

At the end of the activities, each pair summarised, in writing, their experiences with GeoGebra and graphs of modulus functions. These summaries were complemented with notes from whole group discussions held at the end of activity 1. After the first activity, I also led a whole group discussion to summarise some of the observations about the software and the subject content, which had emerged during the activity. Participants expressed these observations in their own language. Table 4.2 summarises the six participants' reflections and experiences about GeoGebra (the software) and modulus functions (the subject content).

Table 4.2 is a summary of the discussions.

<i>Pair</i>	<i>About GeoGebra</i>	<i>About Modulus function graphs</i>	<i>Commentary</i>
<i>Susan and Peter</i>	<ul style="list-style-type: none"> - We had a few problems finding the correct language, but once we started, it was easy to use - It allowed us to complete the 12 tasks easily - The presentation of the function and the graph on the same screen made it easier to understand what graphs of modulus function looked like 	<ul style="list-style-type: none"> - The graph of $y = f(x)$ for linear functions is V shaped - The graph does not exist below the x-axis - You can sketch it by reflecting all parts of $y = f(x)$ below the x-axis in the x-axis - The graph of modulus function $f(x) = x$ is similar to the graph of $f(x) = x$ except that the negative half of the graph is reflected over the x-axis 	<ul style="list-style-type: none"> - The descriptions were not as formal as defined in textbooks. Each pair used different terms to describe the same aspect of graphs of modulus functions. Definitions were based on properties observed on the screens. - There are elements of participants linking their observations to previous knowledge, i.e. a link between the graph of $y = f(x)$ and $y = f(x)$; a discussion about domain and range of functions. - Use of different terminology was used by each group to define the same properties, highlights the aspects of differentiated learning facilitated by GeoGebra.
<i>John and Emma</i>	<ul style="list-style-type: none"> - Easy to use once you know how to input functions, - Colourful displays made it easier to link the graphs, - Could try many functions in a short space of time, - Could try more complicated functions without fear of getting wrong diagrams, - Allowed us to spend more time on understanding the graphs. 	<ul style="list-style-type: none"> - The graph of $y = f(x)$ is symmetric about the x coordinate of the vertex, - The parts of the graph of $y=f(x)$ below the x-axis were reflected in the x-axis, - The graph of $y= f(x)$ does not exist below the x-axis. - The graph of $f(x) = x$ looks like a V with its vertex at (0, 0). Its gradient is 1 on the right side of the vertex, and -1 on the left side of the vertex. 	
<i>James and Sophie</i>	<ul style="list-style-type: none"> - Easy to use, - Clear graphs, displayed on a grid, - Could personalise graphs easily by changing colour and scale, - Could drag or animate functions, - Fast to use and this enabled us to work through many examples in a short space of time, - The multiple representations made it easier to compare graphs by drawing them on the same screen. 	<ul style="list-style-type: none"> - Graph of $y = f(x)$ has some sharp corners, - Graph of $y = f(x)$ does not exist below the x-axis, - To sketch the graph of $y = f(x)$, you can sketch the graph of $f(x)$ first and then reflect every part below the x-axis in the x-axis, - The domain remained the same but the range of functions was reduced to $f(x) \geq 0$. 	<ul style="list-style-type: none"> - Participants were not restricted to textbook definitions but discussed properties as they saw them on the screen.

Most of the descriptions in Table 4.2 were based on the six participants' visual interpretations of the multiple representations. Though all participants were investigating the same concept, I noted that the differences in the language, the phrases in the descriptions and discussion points, varied from pair to pair. Susan and Peter observed that the graph of $y = |f(x)|$ is V shaped and turns at the x-axis, while Emma and John

used the term 'symmetric at the vertex' to define the shape of the graph of $y = |f(x)|$. Emma and John went further to state the equation of the line of symmetry as the x coordinate at the turning point. At this early stage, participants were already beginning to note some properties of the modulus function graphs before the actual activities on modulus functions begins. In their dialogue, John and Emma mentioned two properties: (i) the graph has sharp corners; and (ii) domain and range to compare the graphs of $y = f(x)$ and $y = |f(x)|$. Generally, all pairs used the phrase 'reflection in the x -axis'. It was generally agreed that, to sketch the graph of $y = |f(x)|$, you needed to sketch the graph of $y = f(x)$ first. Then take any part of the graph below the x -axis and reflect it in the x -axis.

Three benefits of using GeoGebra observed at this stage were: (i) increased speed and efficiency (participants worked with accurate graphs, not sketches); (ii) supporting processes of checking, re-trialling and refinement (Ruthven et al 2008). Emma used GeoGebra to support her argument: Fig 4.7) and (iii) enhancement of participants' sense of capability (participants ventured beyond basic algebraic functions to include trigonometry and exponential functions, moving towards creating meaning for what modulus function is about).

This first part of the session was to become a reference point for participants. A set of relevant tools, based on their use in investigating modulus functions, was established.

4.1.2 Activity 2:

Using GeoGebra to investigate the graphs of $y = |ax + b| + c$:

The objective of Activity 2 was to formally start investigating functions of the nature $y = |ax + b| + c$ using sliders, with the intention of leading participants to design interactive applets in GeoGebra.

In order to investigate graphs of the nature $f(x) = |ax + b| + c$ without having to constantly type in new functions in the Input window, it was necessary to introduce the **Slider** tool. The version of GeoGebra used for this study only recognised an input in the format $y = \text{abs}(ax + b) + c$ as correct, if parameters a , b and c were pre-defined as Sliders (see Fig 4.9). A Slider is a tool in GeoGebra that represents a variable. It is a

tool used in interactive and dynamic mathematical learning environments, in the same mode as dragging in Dynamic Geometry or Cabri (Lingguo and Selcuk 2010).

Participants were required to observe and note changes on the graphs of modulus functions, as they changed values of parameters on the sliders. Participants examined graphs in the graphics screen and their corresponding representations shown in the algebra screen and made inferences about the dynamic changes they observed. Colour codes continued to provide a clear link between graphical and algebraic representations on the multiple screens. Motion simulated on the screens afforded participants opportunities to see dynamically, algebraic and graphical changes in real-time.

Activity guidelines:

- *Follow instructions and create Sliders a, b and c.*
- *Enter the function $y = \text{abs}(ax + b) + c$*

Using parameter a

- *Change the value of parameter a by moving the bubble on the slider with a mouse.*
- *How does it influence the algebraic function in the algebra section?*
- *What happens to the graph when you change the value of a?*
- *Write down your observations.*

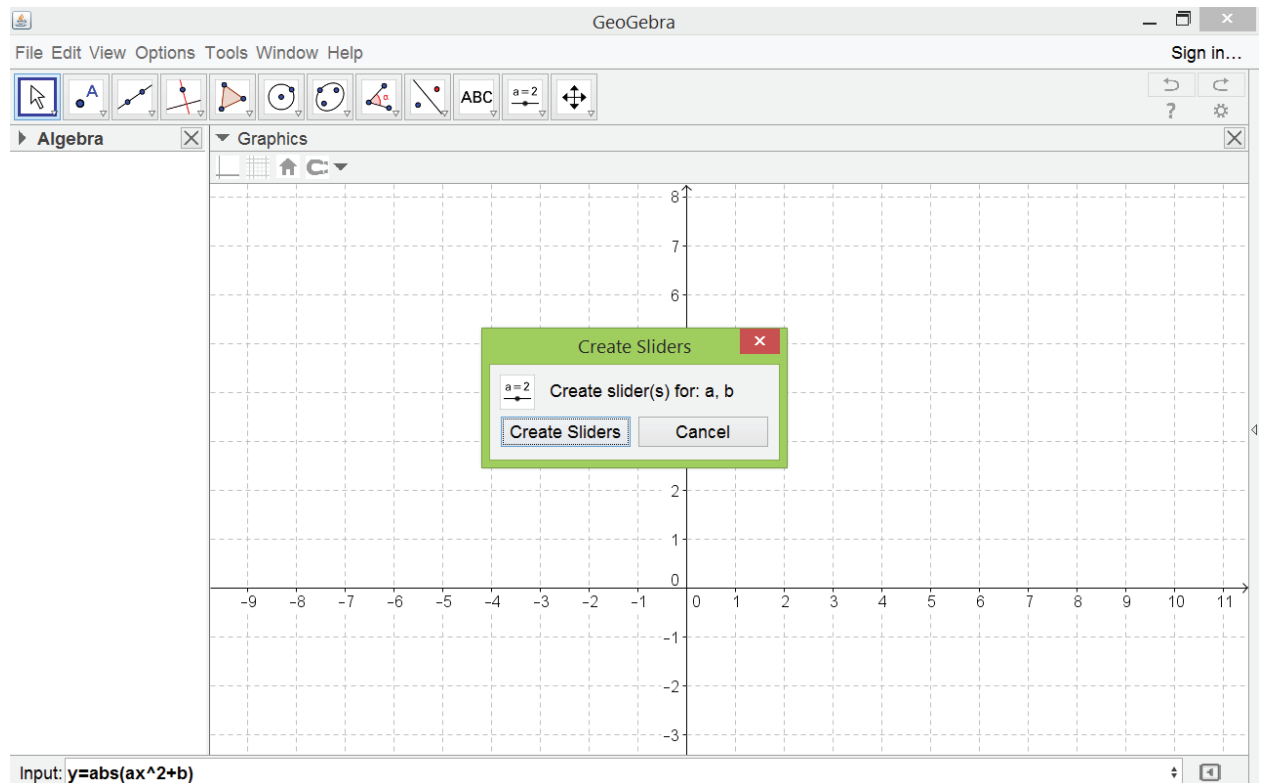
Using Parameter b

- *Change the parameter value b.*
- *What happens to the graph when you change the value of b?*
- *How does this influence the graph of the modulus function?*
- *Write down your observations?*

Using Parameter C

- *Change the parameter value of c.*
- *What happens to the graph when you change the value of c?*
- *What is the relation between the y-coordinate and the x-coordinate of each point of intersection as the parameters change?*
- *Animate all sliders and observe the movements in the Graphics window.*

Fig 4.8: *Creating sliders using GeoGebra Version 5.0:*



Vignette 7 is an extract from a screencast video clip where participants were creating and using Sliders.

Vignette 7: *Getting familiar with sliders:*

James and Sophie

.....

Line 181: *Selects the ‘Slider’ icon on the tool bar;*

Line 182: *A click in the graphics section of the screen;*

Line 183: *A dialog window appears (See Fig 4.9);*

Line 184: *A click on ‘Apply’;*

Line 185: *Slider labelled a appears in the graphics section of the screen (Fig 4.10);*

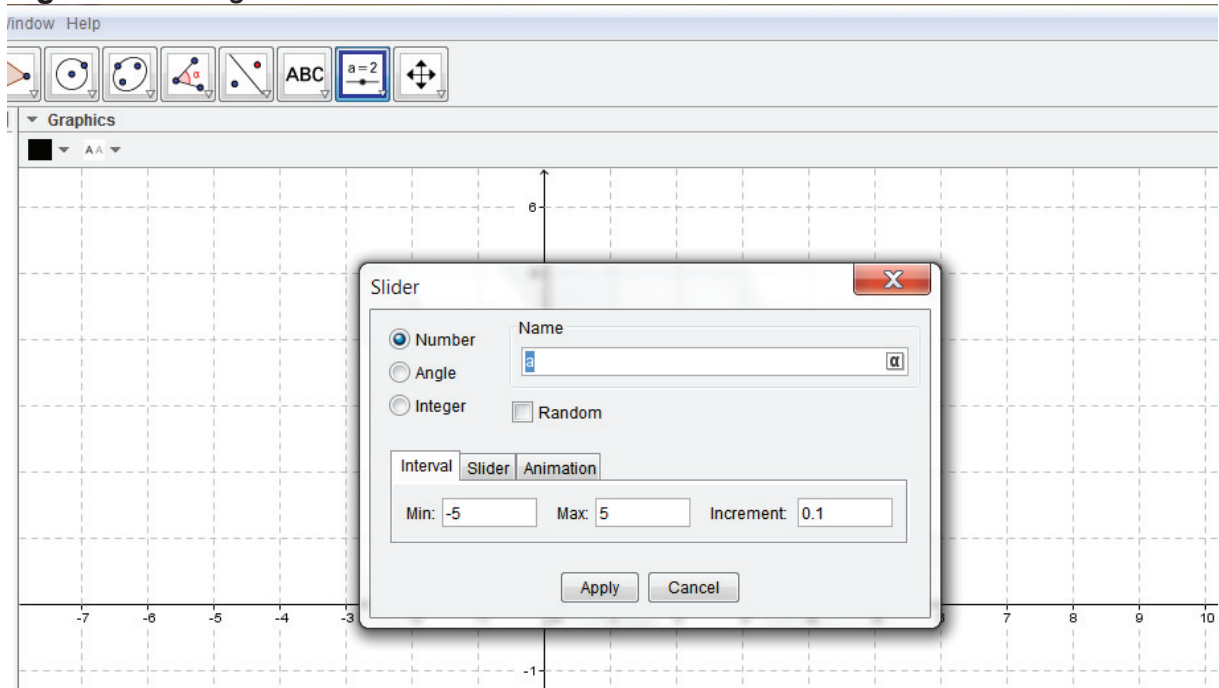
Line 186: *The process is repeated (a slider labelled b automatically appears on the screen,*

Line 187: *..... another slider is automatically labelled c (See Fig 4.10,*

Line 188: *Someone types $y = \text{abs}(ax + b) + c$ in the INPUT box,*

Line 189: *A graph appears in the graphics view (See Fig 4.8).*

Fig 4.9: Creating Sliders:



This is the screen that appears after Line 183.

Fig 4.10: This is the screen that appears after lines 185 – 187.

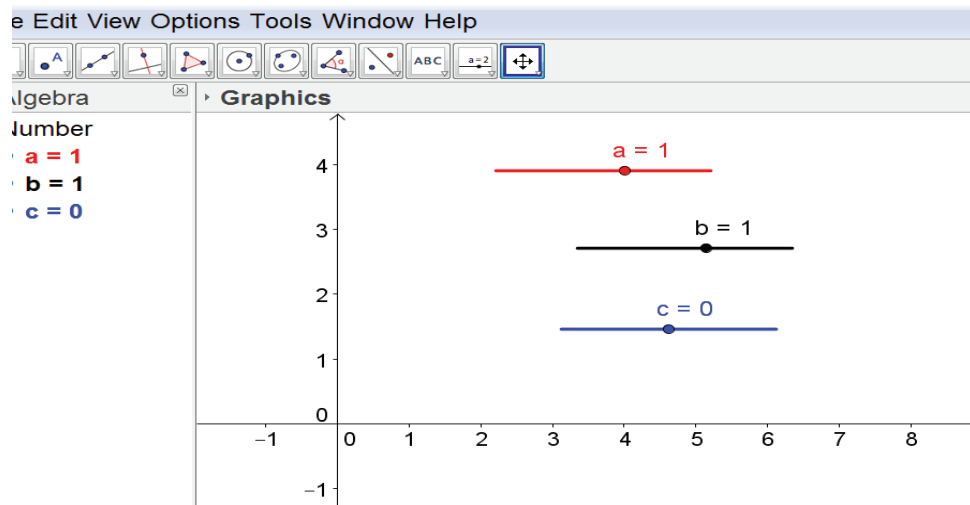


Fig 4.11: Graph of $y = |ax + b| + c$ (Line 188-189).

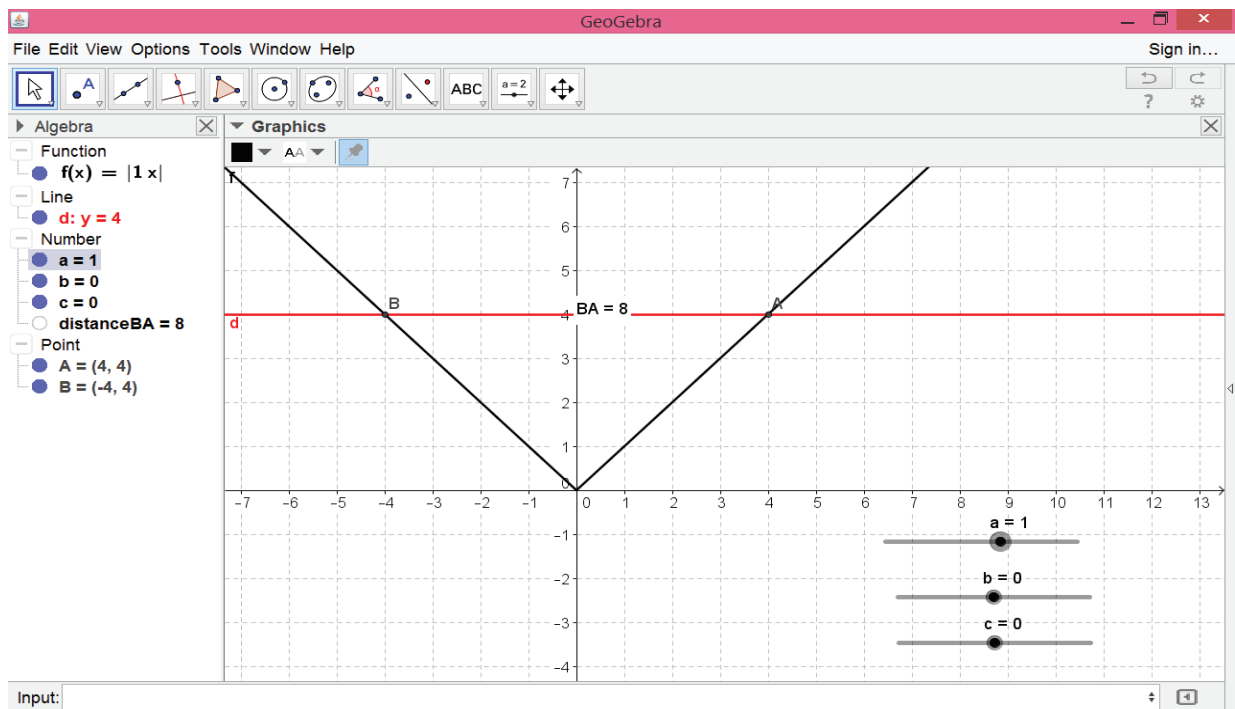


Fig 4.11 shows the graph of $y = |ax + b| + c$ after the creation of the sliders. Fig 4.11 was a construction that changed in real time when the participants changed values on the sliders. By increasing the value of 'a', participants observed the V shaped graph of $f(x) = |ax + b| + C$ becoming narrower.

Vignette 8: *The effects of changing the values of slider $y = |ax + b| + c$.*

James and Sophie

.....

Line 319: *Moves the slider b to the right and the graph moves to the left*

Line 320: *As b increases, the graph is translated to the left, and moves to the right when b decreases,*

Line 321: *The distance AB remains unchanged.*

Line 322: *Sophie says, "Translation vector is $\begin{pmatrix} -b \\ 0 \end{pmatrix}$ ".*

Line 334: *As c changes value, the graph moves vertically up or down the screen.*

Line 335: *The distance AB changes, as value on slider c is changes.*

Line 336: *James remarks, "There is a vertical translation vector $\begin{pmatrix} 0 \\ c \end{pmatrix}$ ".*

Line 336: *Sophie: "When c is greater than four, the distance AB is also undefined".*

Line 338: *Sophie: "... but when c becomes smaller than 4, the distance AB continuous to be displayed as zero"*

GeoGebra has a tool that measures distance between two defined points. However, this tool has its shortcomings as observed in the video clip. In the process of adjusting Slider value of ' c ' to translate the graph of $y = |ax + b| + c$ above the line $y = 4$, distance AB is specified as 'undefined' on the screen. This message 'Distance $AB = 0$ ' remains on the screen even after the graph has been translated back to its original position. GeoGebra ceases to measure the correct distance between points A and B . To continue with the investigation, James and Sophie had to clear the screen and start the whole process in Vignette 8 again.

James and Sophie concluded that the graph of the modulus function $f(x) = |ax + b| + c$ is a V-shaped with vertex $(-\mathbf{b/a}, \mathbf{c})$ and gradient ' \mathbf{a} ' on the right side of the vertex ($x > -b/a$) and gradient ' $\mathbf{-a}$ ' on the left side of the vertex ($x < -b/a$). This observation sits well for linear functions involving the modulus function, but more investigations for polynomials of degree two or higher involving modulus functions needed to be done.

Susan and Peter settled for the quadratic function $f(x) = |ax^2 - b| + c$. Vignette 9 is based on their activity as they investigated the properties of the function. They created

an animated applet to investigate the effect of sliders on a quadratic graph involving modulus functions.

Fig 4.12: Screencast shot from James and Sophie's animated applet.

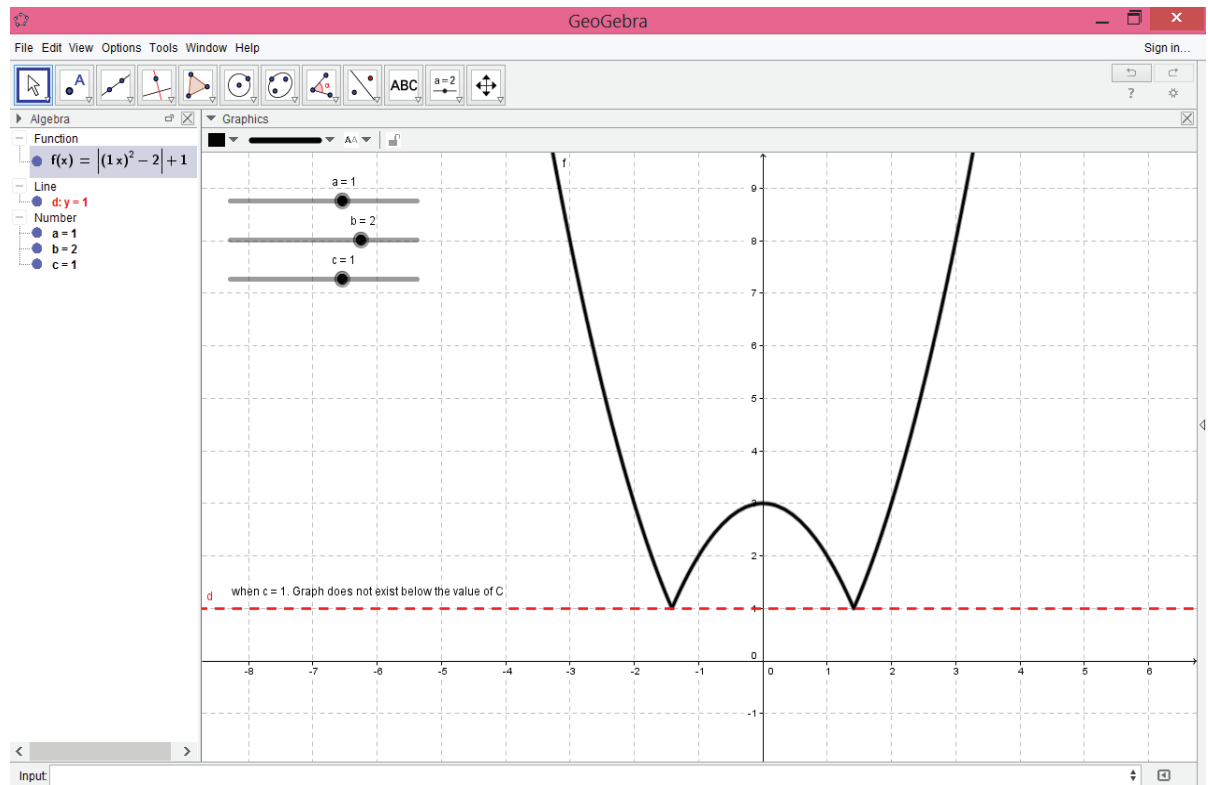


Fig 4.12 is a screencast shot from an animated applet created by Susan and Peter.

The Vignette 9 started after all the processes of creating an applet had been completed.

Vignette 9: *Further investigations using animated applets:*

.....

Susan and Peter

Line 363: Peter: “As ‘a’ moves to the right, the graph becomes narrower. It becomes wider as ‘a’ moves to the left”.

Line 364: Susan, “Let’s animate Slider ‘b’”.

Line 365: Click on slider ‘b’,

Line 366: Selects ‘Animation on’,

Line 367: Peter: “Shape of the graph changes as the bubble on the slider moves from -5 to 5”;

Line 368: Susan: “Describe how”;

Line 369: Peter: “This one is difficult to describe. At one point it becomes a U shape and at another point, I can see the ‘mount’ in between becoming bigger and bigger”.

Line 370: Slider ‘c’ is set on ‘animation on’ and the graphs moves up and down the screen;

Line 371: Susan briefly switches off the animation [Pause],

Line 372: Peter says, “The graphs seem to turn at whatever value of ‘c’ we set. Just check... If you change $c = 1$, the graph does not go below the line $y = 1$ ” (See Fig 4.12).

Line 373: Value of c is changed on several occasions,

Line 374: Susan: “It’s true. The value of c determines the minimum points on the graph”

Line 375: Peter: “The range of each function is for values of $f(x) \geq c$ ”.

Line 376: Susan: “Yes, the graph is always reflected in the line $y = c$ ”.

There was an instance when Sliders ‘a’, ‘b’ and ‘c’ were animated and the graph moved around the screen but remained above $f(x) \geq c$. It was difficult to extract much conversation from the video clip because in most instances there was no audible conversation recorded during animations on the applet. However, there was evidence of participants beginning to formulate general descriptions of the behaviour of the graph of $y = |ax^2 - b| + c$ (see lines 375 and 376). Vignette 9 gives some evidence of the internalisation of the process of using sliders and the slider artefact becoming an instrument. Internalisation of the slider tool is depicted in the way it is used in subsequent activities to solve new problems.

Besides illustrating the potentialities of the slider tool as a semiotic mediator, the work also showed the contribution of this tool to the emergence of a definition of the properties of graphs of modulus functions. The discussion in Vignette 9 illustrates meaning emerging from the activity using sliders. Susan and Peter tried to make certain interpretations by moving the sliders and analysing changes that occurred in the multiple representations, as viewed in the algebraic and graphics windows.

Table 4.3 summarises observations from screencast video clips and conversations captured during activity 1 and 2. The summary avoids duplicating observations common to all the three pairs. It only highlights observations that were unique to each pair.

Table 4.3: A summary of participants' observations.

Participants	Observations	General Commentary
John and Emma (Using the quadratic function)	<ul style="list-style-type: none"> - When slider 'a' changes, there is a horizontal stretch with stretch factor $1/a$ - The graph never goes below the x-axis - When a is positive, the graph assumes the shape of a quadratic function, When a is negative, part of the quadratic graph that would have been below the x-axis is reflected in the x-axis 	<ul style="list-style-type: none"> - The use of GeoGebra generated a lot of discussions as participants took time to observe the animated applets without manually entering any data.
Susan and Peter (Using a linear function)	<ul style="list-style-type: none"> - The graph is wider than the graph of $y = x$ if $a < 1$ and narrower than the graph of $y = x$ if $a > 1$ by stretch factor $\frac{1}{ a }$. - The graph always has a vertex at $(-b, c)$ and is symmetric in the line $x = -b$ - The graph is V-shaped. It opens up if $a > 0$, straight line when $a = 0$ and opens up again for $a < 0$. 	<ul style="list-style-type: none"> - Some of the notation in this table has been modified to make the thesis look better.
James and Sophie	<ul style="list-style-type: none"> - Though using a quadratic function, James and Sophie noticed that 'a' stretches the function horizontally. - Parameter 'b' moves the function up and down the screen - Parameter 'c' sets the range of the function [they did not use the word range, but described it as the limit beyond which the graph does not exist] 	<ul style="list-style-type: none"> - Though working on different screen, and using different descriptive language, all pairs seemed to come to the same general conclusions about the transformations

Hershkowitz et al (2001) view generalisations as an activity of reorganising previously constructed mathematical knowledge into a new mathematical structure. Instances of generalisations arising from recognition of commonalities are summarised below. However, it was difficult to deduce from conversations whether participants really understood the implications of the dynamic changes beyond the animations on the screen.

The six participants engaged in some mathematical discourse, explicitly formulating the general observations from a collection of activities carried to date. Falcade et al (2007) argue that collective discussions play an essential part in teaching and learning and constitute the core of semiotic process on which learning is based. After the activities, various observations were collectively discussed, collectively analysed and commented on. There are common points noted from the summary of observations in Table 4.3 regarding characteristics of the graphs of $y = |ax + b| + c$ and $y = |ax^2 - b| + c$.

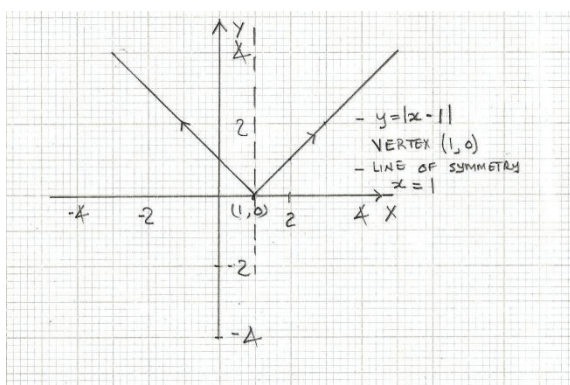
- The graph of $y = |ax + b| + c$ has a vertex at $(-b/a, c)$ and is symmetric about the line $x = -b/a$.
- The parameter 'a' stretches the graphs and the graphs are wider for values $-1 < a < 1$ and narrower for values $|a| > 1$.
- Both functions are horizontal and linear for value of $a = 0$.
- The linear graphs involving modulus functions are V-shaped. They open up if $a > 0$ and $a < 0$.
- Parameter c sets the lowest limit of the graphs of modulus function.

Though the work in Activity 2 focused on investigating graphs of modulus functions, there were also some elements of transformation of graphs that featured in participants' conversations (see Table 4.3).

4.1.3 Activity 3

Activity 3 was the last activity in this session. Some of the tasks set in this activity aimed at checking if participants had developed mental images of the graphs of modulus functions. Participants used pencil and paper first. They only used GeoGebra to check their pencil and paper work answers. The general objective of the activity was to check if participants could apply knowledge acquired from previous activities without the use of GeoGebra. Participants were asked to draw the graph of $y = |f(x)|$ from the graph of $y = f(x)$. Participants were required to explain their answers with the aid of sketches. Sketch 4.1 and 4.2 are extracts from the six participants' written work.

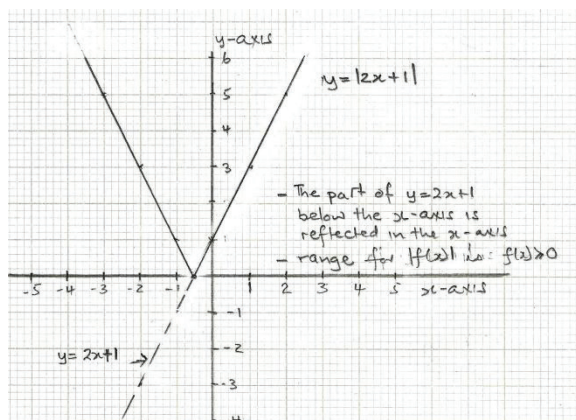
Sketch 4.1: From John and Emma



Sketch 4.1 is a sketch of the graph of $y = |x - 1|$ taken from John and Emma's pencil and paper work. The diagram shows the graph of $y = |x - 1|$. There was no evidence to indicate that this sketch was a product of group work from the pair. However, there were extracts from the John and Emma's discussion scripted on the same page as the graph

claiming that the vertex is at $(1, 0)$; line of symmetry is $x = 1$ (See Sketch 4.1).

Sketch 4.2: Susan and Peter



Sketch 4.2 shows the graph of $y = |2x + 1|$ superimposed on the graph of $y = 2x + 1$. A few notes were scribbled on the graph describing how the pair decided what the graph looked like ("the part of $y = 2x + 1$ below the x-axis is reflected in the x-axis. Range for the modulus function $|f(x)| \geq 0$). The graph has two parts. The right hand side has equation $y = 2x + 1$ and the left hand side has equation $y = -2x - 1$.

The pencil and paper sketches show that the participants understood the relationship between graphs of $y = f(x)$ and $y = |f(x)|$ for simple linear functions. General knowledge of graphs of modulus functions acquired during activities 1 and 2 was demonstrated when participants sketched graphs in activity 3. Participants recognised that (i) the graph of $y = |f(x)|$ is V shaped; (ii) the part of the graph of $y = f(x)$ below the x-axis is reflected in the x-axis and (iii) the graph of $y = |f(x)|$ does not exist below the x-axis. The participants considered the part of the graph below the x-axis and reflected it in the x-axis. There was evidence from the sketches that participants managed to sketch linear modulus functions of the nature $y = |x - 1|$ without using GeoGebra. Participants used GeoGebra to verify the pencil and paper sketches.

To conclude activity 3, participants got an exercise consisting of graphs of modulus functions. In this activity, participants were required to identify and write down the algebraic functions represented by each graph (See Appendix E). The exercise consisted of linear graphs. All six participants correctly matched the functions and their corresponding graphical representations. This task was a simple recognition task, not one of construction or of reading coordinates of points. Duval (2006) argues that, naturally if students are given the expressions to sketch the graph, the success rate would exceed ninety per cent but the results will be different the other way round.

Results from participants' work support Ainsworth's (1999) claims that providing learners with a rich source of representations helps them to translate or construct

references across these representations. The use of GeoGebra and its multiple representations facilities in previous activities provided learners with clear links between the functions and their graphical representations and this was evident in the way they answered questions in the exercise.

My role throughout the first session was to encourage the use of GeoGebra while participants were working on tasks related to modulus functions. I guided participants through basic operations of GeoGebra and helped with some mathematical interpretations where necessary. I also guided participants through the multiple screens of GeoGebra and explained where to find the relevant tools, how to use colour codes and how to enlarge the scales using the mouse. The availability of an Interactive Whiteboard allowed me to give group instructions whenever it was necessary.

4.2 SESSION 2

Polynomials, exponentials and trig functions involving modulus functions.

Session 2 is divided into three activities. In the second session, participants used both GeoGebra and calculus to investigate graphs of polynomials involving modulus functions. Activity 1 focused on the graphs of $y = |f(x)|$ for quadratic functions ($y = |ax^2+b|$), trigonometric functions, exponentials and logarithms. Participants also considered the use of calculus to find derivatives of modulus functions. Activity 2 focused on polynomials of degree higher than 2 that involved modulus functions. The last activity focused on the use of the 'tangent' tool on the GeoGebra tool bar, to investigate the aspects of continuity and differentiability of modulus functions.

The Core 1 module in the 'A' level mathematics curriculum required students to be able to use the method of *completing the square* to identify the vertex and roots of quadratic functions. Graph sketching is further enhanced in the Core 2 module when students use differentiation to identify stationary points using the fact that the derivatives of functions are zero at stationary points.

When confronted with the graph in Fig 4.12 Peter and Susan used calculus to identify the stationary points of the function.

4.2.1 (a) Activity 1:

Fig 4.13: A copy of Fig 4.12.

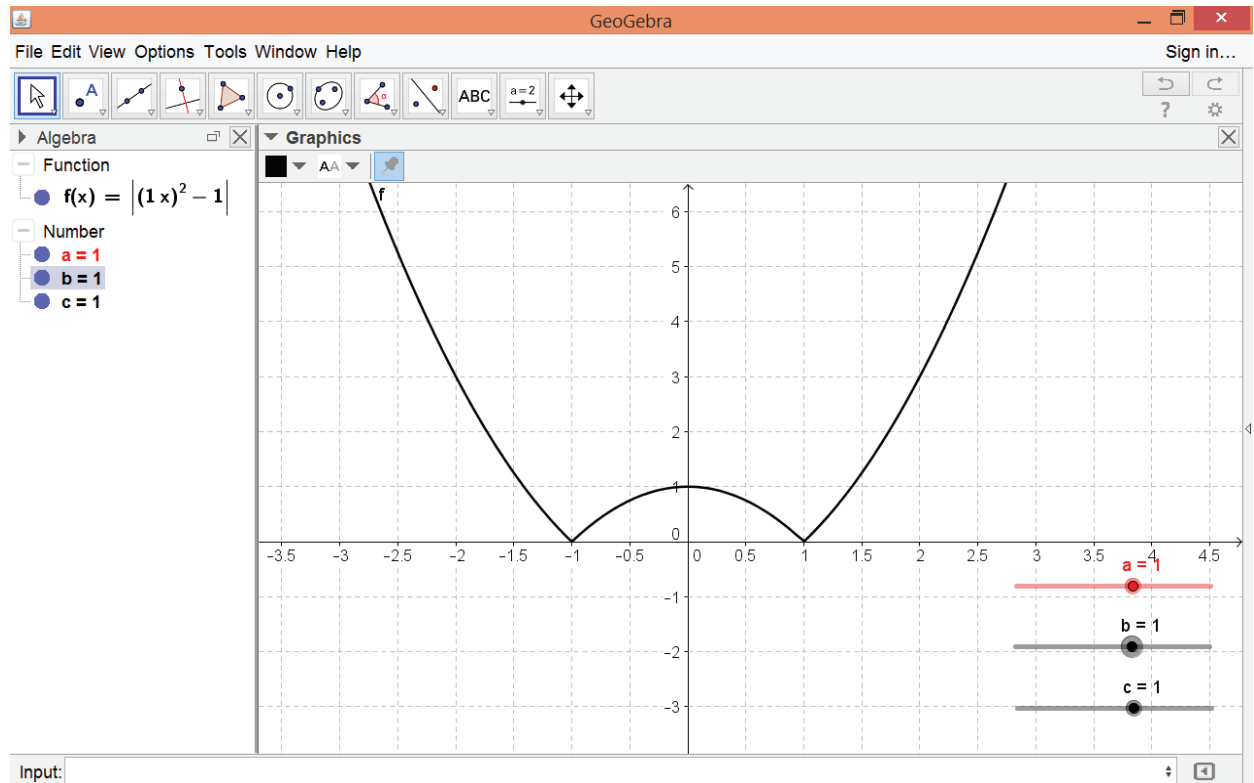


Fig 4.13 is a transformed version of Fig 4.12. In this phase Susan and Peter tried to use calculus to identify turning points of the graph of $f(x) = |x^2 - 1|$ (see Insert 4.1 below). They observed that the graph of $y = |f(x)| = |x^2 - 1|$ had vertices at $(-1, 0)$ and $(1, 0)$ and a local maximum at $(0, 1)$ when all slider values are set at 1. The pair regarded all the three as stationary points. Their curiosity was to check if these vertices and the local maximum could be identified using calculus. GeoGebra provided a visual representation (Fig 4.13) showing turning points. Susan and Peter concluded that the function could therefore, be differentiated to identify the stationary points.

Susan and Peter used differentiation to identify stationary points on the graph of $y = |x^2 - 1|$. They differentiated the modulus function. The work represented below as an insert, is work extracted from Susan and Peter's pencil and paper work. Below is an extract from Susan and Peter's use of calculus to investigate stationary points on the graphs in Fig (4.13).

Insert 4.1

$$y = |x^2 - 1|$$

$$y^2 = (x^2 - 1)^2$$

$$2y \frac{dy}{dx} = 2x \times 2(x^2 - 1)$$

$$\frac{dy}{dx} = \frac{4x(x^2 - 1)}{2|x^2 - 1|}$$

$$\frac{dy}{dx} = \frac{2x(x^2 - 1)}{|x^2 - 1|}$$

$$\text{When } \frac{dy}{dx} = 0$$

$$2x(x^2 - 1) = 0$$

$$x = 0; x = 1 \text{ or } x = -1$$

From previous knowledge of graphs of functions, Susan and Peter concentrated on the critical features of a graph (i.e. maxima, minima and intercepts). The handwritten work above indicates that the pair equated the derivative to zero, and obtained three values of x , $x = -1$; $x = 0$ and $x = 1$. Susan and Peter were of the view that there are three stationary points for the function $y = |x^2 - 1|$. However, John reminded Susan and Peter that the values of x could not be 1 or -1 since the denominator in the derivative would be zero and the derivative would be undefined. The written work, where Peter and Susan equate the derivative to zero, shows a misconception that all turning points are stationary.

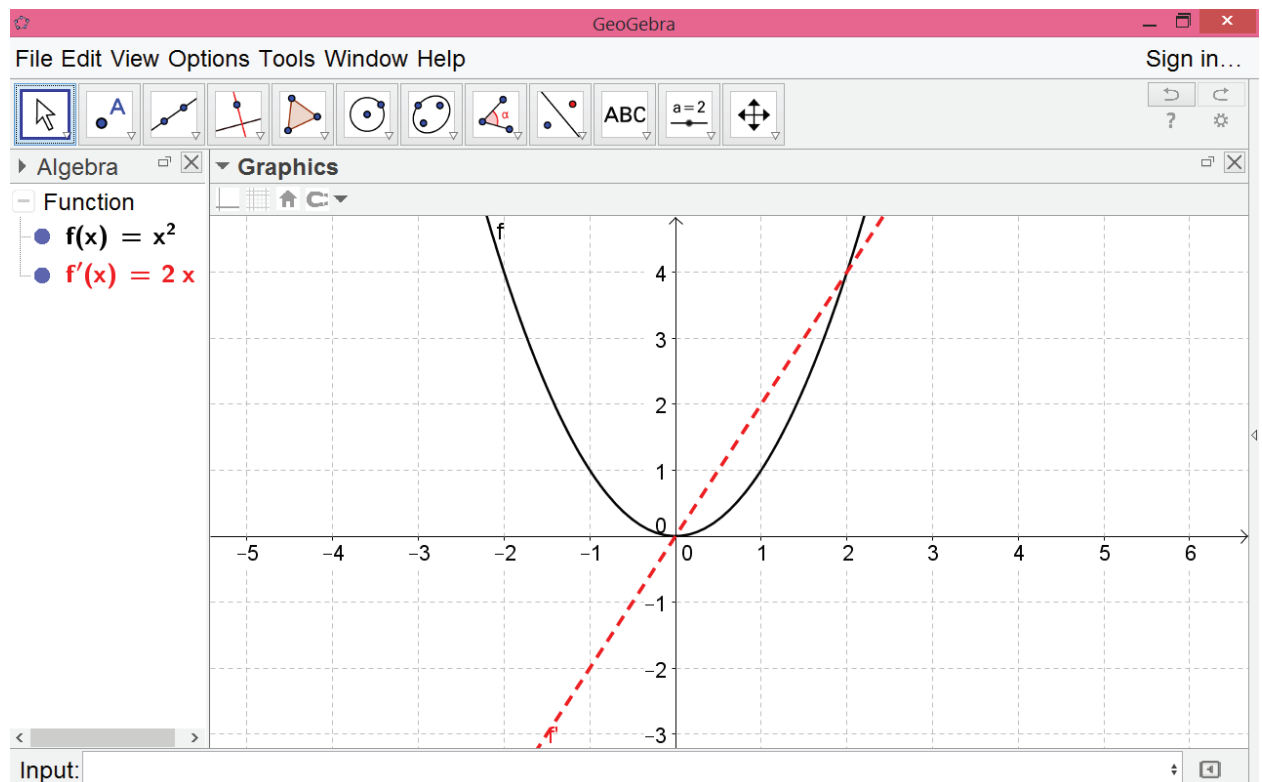
4.2.1 (b) Using GeoGebra to verify the derivative of $f(x) = |x^2 - 1|$.

GeoGebra has a facility for finding derivatives of given functions. I used the Interactive Whiteboard to demonstrate how to enter the correct syntax in the INPUT window. Initially I used simple functions like $f(x) = x^2$, $g(x) = \ln(x)$ and $h(x) = \sin(x)$ to demonstrate

how to differentiate functions using GeoGebra. The functions were selected from work that participants had encountered previously in calculus, hence their derivatives were familiar. Participants were however not familiar with the graphical representations of the derivatives.

Once a function $f(x)$ or $g(x)$ was defined, typing " $f'(x)$ " or "Derivative $[f(x)]$ " in the input window and pressing ENTER key yielded the derivative function in the algebraic window and its graphical representation in the graphics window. Participants received instructions on how to use GeoGebra to differentiate $f(x) = x^2$, $g(x) = \ln(x)$ and $h(x) = \sin(x)$. Fig 4.14; Fig 4.15 and Fig 4.16 are three extracts from participants' work.

Fig 4.14: The graph of $f(x) = x^2$ and its derivative $f'(x) = 2x$.



Graphical representation of $f(x)$ clearly shows the graph of linear function $f'(x) = 2x$. Participants were already familiar with the graphs of $f(x) = x^2$ and $f'(x) = 2x$. The video clip was short indicating that the participants did not spend much time on the displays of these functions. The diagram shows the behaviour of the gradient of the function $f(x) = x^2$.

Fig 4.15: The graph of $g(x) = \ln(x)$ and its derivative.

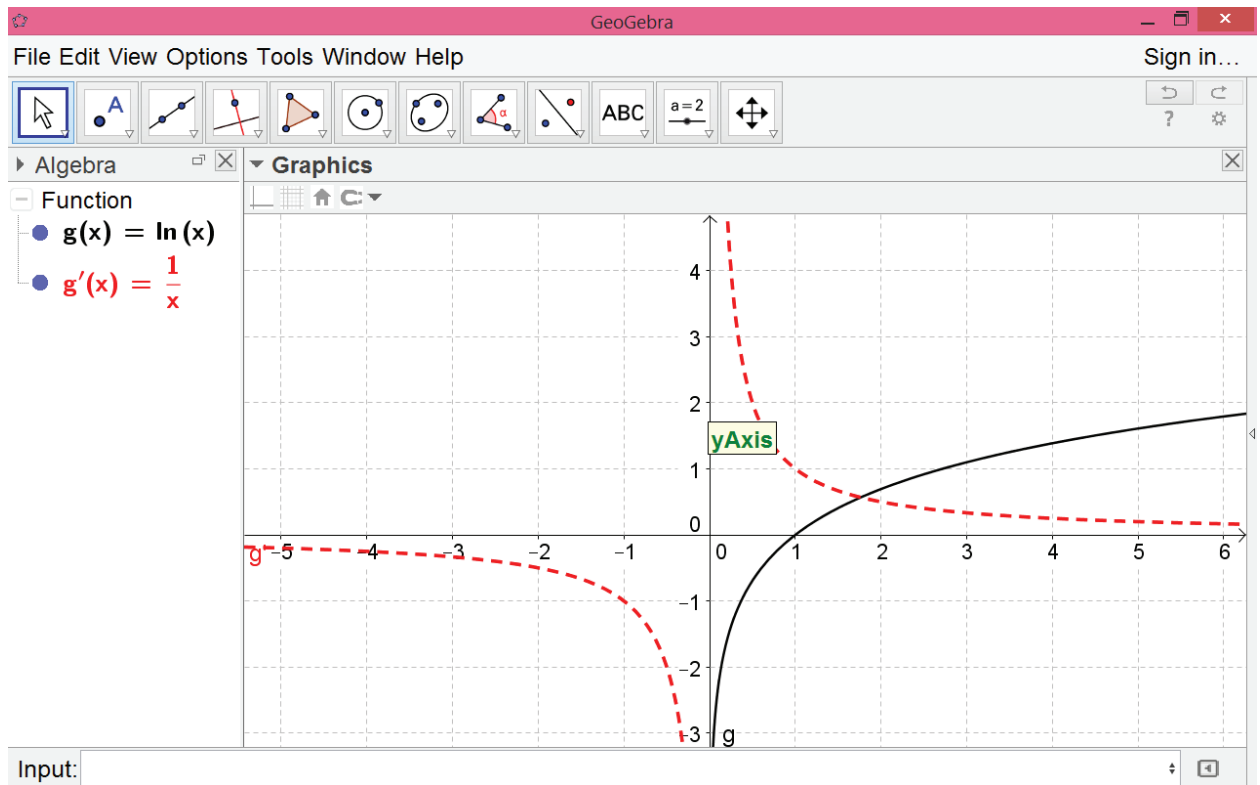
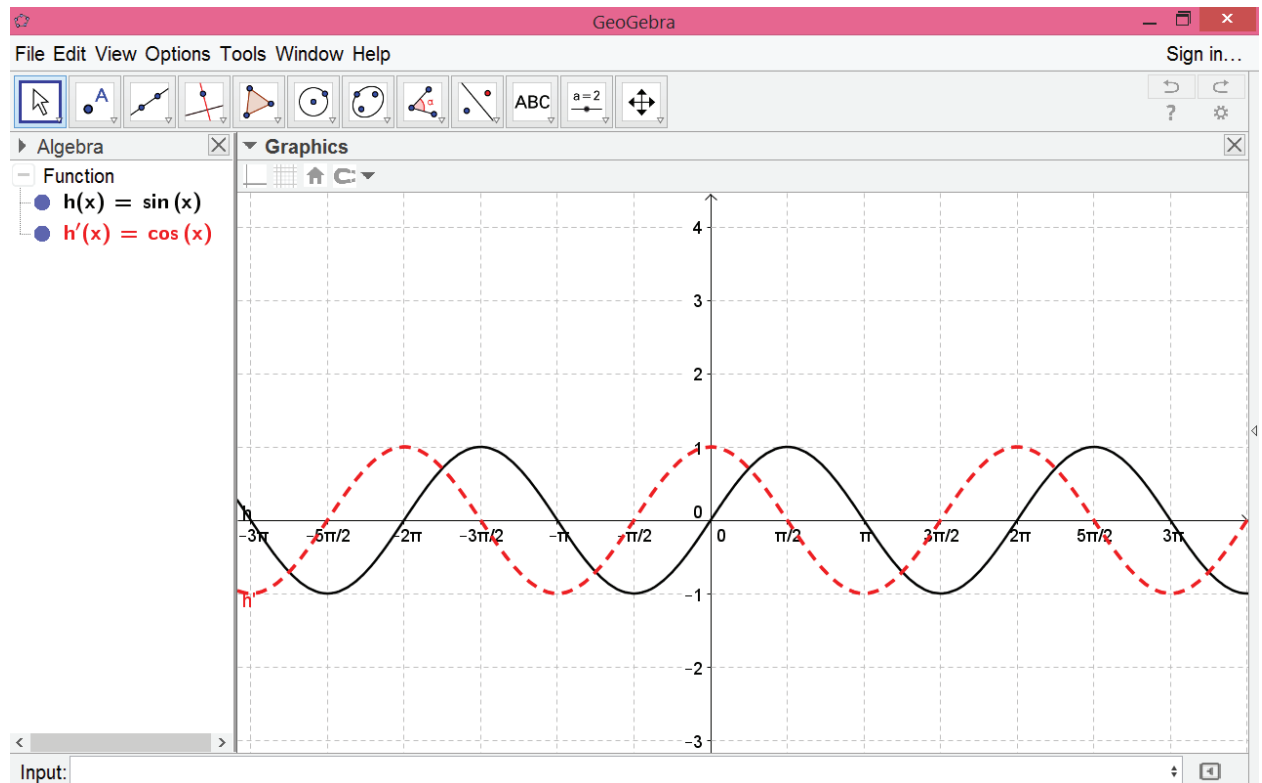


Fig 4:15 displays the graph of $g(x) = \ln(x)$ and its derivative $g'(x) = \frac{1}{x}$ as a rectangular hyperbola.

Fig 4.16: The graph of $h(x) = \sin(x)$ and its derivative $h'(x) = \cos(x)$.



The derivative of $h(x) = \sin(x)$ is represented by the graph of $h'(x) = \cos(x)$.

The work in Figures 4.14 – 4.16 was a precursor to the investigation of derivatives of modulus functions using GeoGebra. The graphical representation of derivatives for $f(x)$, $g(x)$ and $h(x)$ came in as an additional aspect to participants' learning. The current Core 2 and 3 curriculum does not put much emphasis on graphs of derivative functions. However, participants recognised the respective graphs $f'(x) = 2x$; $g'(x) = 1/x$ and $h'(x) = \cos(x)$.

The next Vignette (10) is an extract from an attempt by Susan and Peter to use GeoGebra to differentiate modulus function graphs.

Vignette 10: *Using GeoGebra to differentiate modulus functions:*

Line 578: *Peter types $y = \text{abs}(x^2-4)$;*

Line 579: *A graph appears in the graphic window (see the black graph in Fig 4.16),*

Line 580: *Peter types " $f'(x)$ ",*

Line 581: *The derivative of a function $f(x) = |x^2 - 4|$ appears in the algebraic view as $f'(x) = \frac{2x(x^2 - 4)}{|x^2 - 4|}$, at the same time more lines (later coded with the red colour in Fig 4.17) appears in the graphics view.*

Line 582: *No comment about the representations in the graphics view,*

Line 582: *"Our differentiation was correct", Peter remarks,*

.....

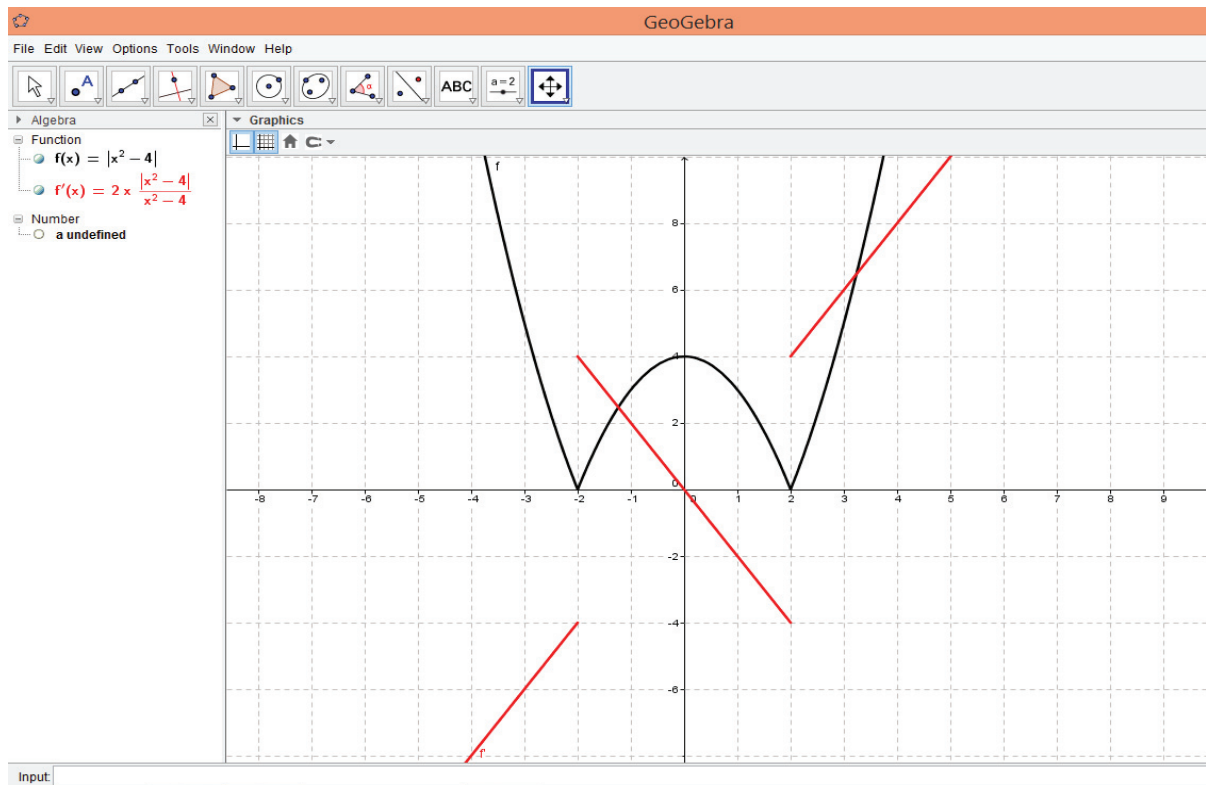
Line 633: *Me: "remember to check the value of the gradient function when $x = 2$ or $x = -2$ ".*

Line 634: *"The derivative cannot be defined at $x = 2$ and $x = -2$ because the denominator will be zero", Peter says.*

Line 635: *"Let us draw tangents through the three points," said Susan,*

.....

Fig 4.17: The function $y = |x^2 - 4|$ and its gradient function.



Two graphical representations for $f(x) = |x^2 - 4|$ and $f'(x) = \frac{2x(x^2 - 4)}{|x^2 - 4|}$ appeared in Fig 4.17. The conversation on the video clip made very little reference to the graph of $f'(x)$ (red function). I interpreted this lack of reference either as an indicator that Susan and Peter did not relate the graphs to any work on modulus functions they had encountered previously or that their focus was on the graphs of $f(x) = |x^2 - 4|$. Susan and Peter verified their pencil and paper answer (Line 582) using the derivative $f'(x)$ function in GeoGebra.

I displayed the graphs of $f(x) = |x^2 - 4|$ and that of its derivative, on the Interactive Whiteboard and colour coded the lines to provoke discussion about the graphical representations displayed on the screen, but with emphasis of the graph of $f'(x)$. After some discussion, the six participants agreed that the graph of $f'(x)$ (in red) in Fig 4.17 illustrated the behaviour of the derivative of $f(x) = |x^2 - 4|$. The discussion points are listed below.

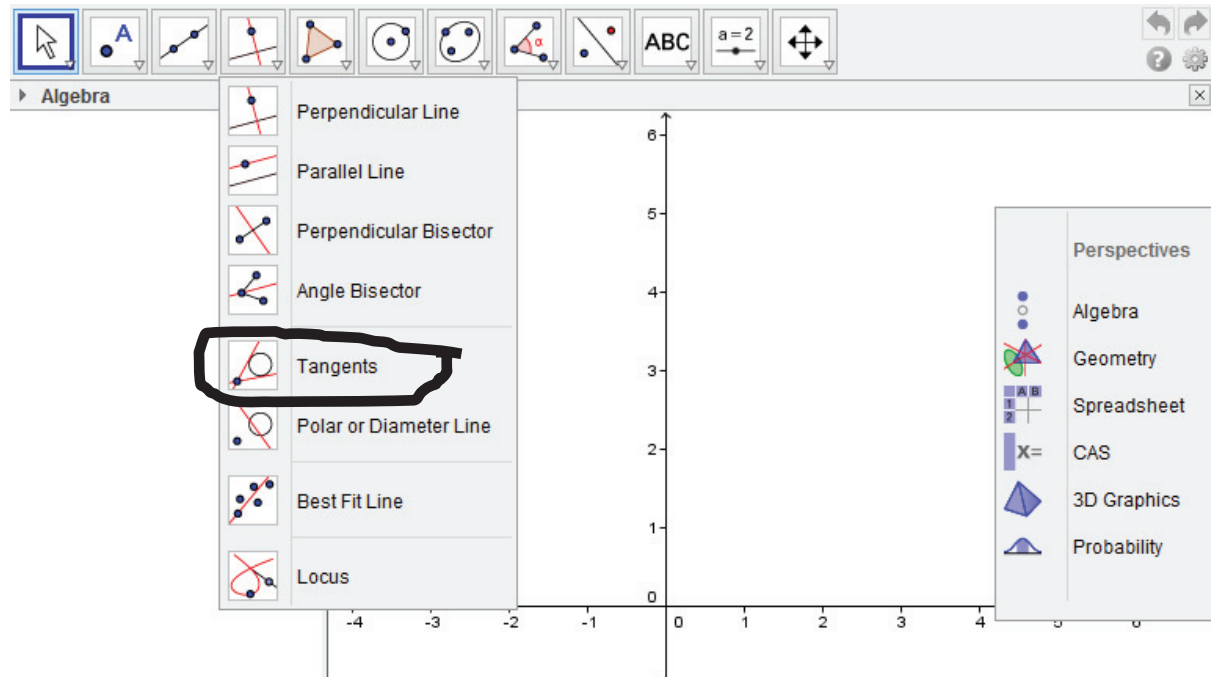
- a. For values of x to the left of -2 , the function is decreasing therefore the red line is found below the x - axis.
- b. At $x = -2$ the graph displays some discontinuity.
- c. The red line appears above the x – axis between -2 and 0 , indicating that the function $f(x)$ is increasing in that interval.
- d. Between 0 and 2 , $f(x)$ is decreasing, and the red graph goes below the x – axis.
- e. The continuity of the red line in the interval $-2 < x < 2$, crossing the x – axis 0 indicates a stationary point when $x = 0$.
- f. In the interval $x > 2$, $f(x)$ is increasing and the red graph is above the x – axis.
- g. The discontinuity of the red graph at $x = -2$ and $x = 2$ implies that the derivative function is undefined at those x values.

4.2.2 Activity 2:

Activity 2 introduces the ‘Tangent’ icon. This tool affords the ability to construct tangents at given points of the function. This was just an extra activity, to demonstrate further that the sharp corners on the graph of a modulus function are not stationary points. The Tangent tool can be found of the GeoGebra menu bar (Fig 4.18).

The use of calculus to identify turning points in this activity brought to fore misinterpretation or misunderstanding of the relationship between stationary and turning points. I reminded participants that tangents through stationary points are horizontal, and that not all turning points are stationary.

Fig 4.18: Using the tangent construction tool to identify stationary points:



Screencast video evidence showed that none of the pairs managed to hold the tangent horizontally at $x = 2$ or $x = -2$. Failure to construct a horizontal tangent at $(2, 0)$ and $(-2, 0)$ further helped to clarify the distinction between stationary points and turning points. This explained why the gradient function was undefined at points $(2, 0)$ and $(-2, 0)$. Participants noted that $\frac{dy}{dx} = 0$ only at stationary points. They also noted that not all turning points are stationary.

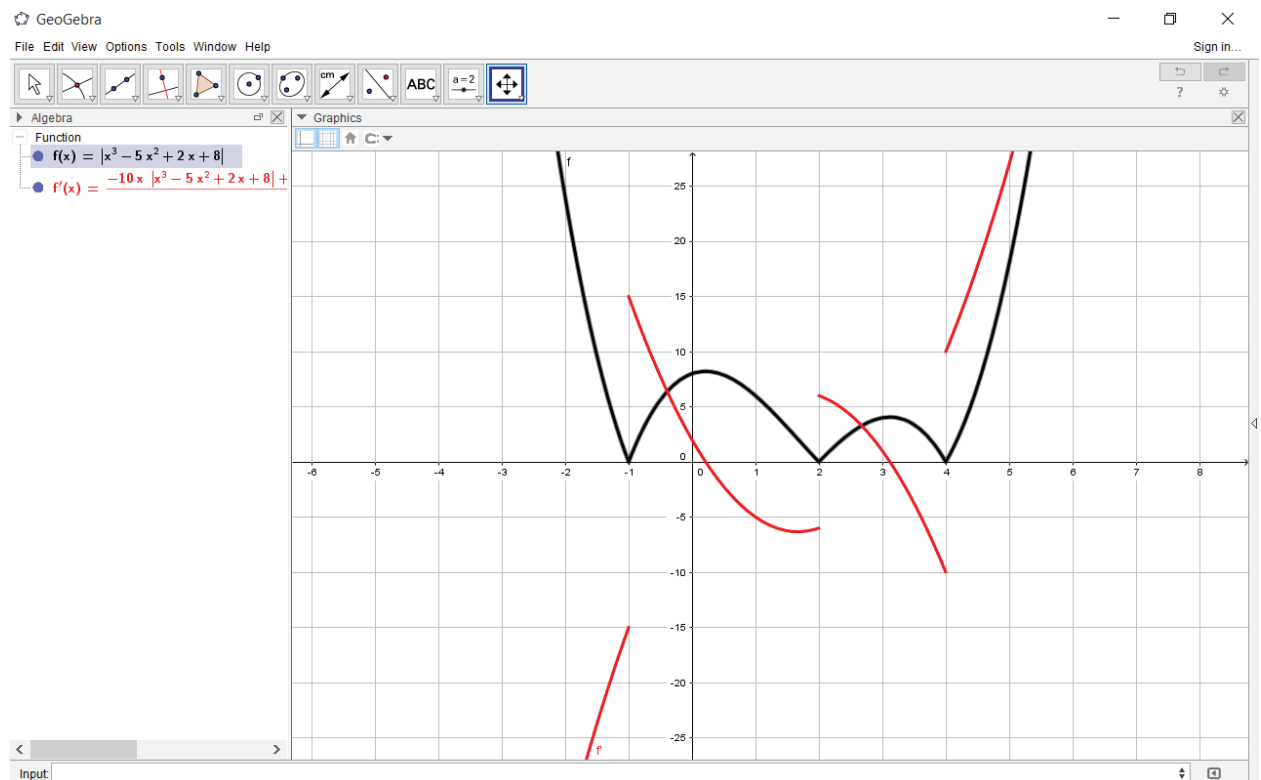
Using calculus to identify turning points in this activity also brought to fore misinterpretation or misunderstanding of the relationship between continuity and differentiability. This activity raised two issues for further investigation: (i) Is a differentiable function always a continuous function. (ii) Is a continuous function always differentiable? This was knowledge emerging outside the concept of modulus functions through working with GeoGebra. Fig 4.17 and Fig 4.19 display graphs, which clearly show sharp turns and some stationary points. This generated discussion among participants on their understanding of differentiability of functions. GeoGebra supported students to go beyond the confines of the problem and provided a facility for extending their knowledge about modulus functions.

Work in this activity showed that participants first used graphical representations in GeoGebra at a very low level to justify the correctness of the pencil and paper work (Line 582). They also used GeoGebra to verify their answer, receiving instant feedback about the correctness of their calculus. At another level, GeoGebra was used to correct the misconceptions about stationary and turning points.

4.2.3 Activity 3

In this activity participants carried out further investigations on polynomials involving modulus functions and their derivatives. The differentiation of polynomials involving modulus functions became too complicated and participants used GeoGebra to find the derivatives. They were now familiar with the process of differentiation using GeoGebra, and did not waste time using pencil and paper first.

Fig 4.19: The graph of $f(x) = |x^3 - 5x^2 + 2x + 8|$ and its derivative.



When John typed $f'(x)$ in the input window, a very long function appeared in the Algebraic View

$$f'(x) = \frac{2|x^3 - 5x^2 + 2x + 8| - 10x|x^3 - 5x^2 + 2x + 8| + 3x^2|x^3 - 5x^2 + 2x + 8|}{x^3 - 5x^2 + 2x + 8}$$

John acknowledged that it would have been very difficult for him to complete the differentiation using pencil and paper.

Work in this activity showed that John and Sophie tried to use GeoGebra for visual justification about graphs of modulus function not existing below the x-axis. However, the multiple screens displayed the visual aspect of the function and its derivative on the algebra screen, and the graph of the modulus function and its derivative on the graphics screen. There were no further recorded discussions to check their understanding of the multiple representations of the derivatives of polynomials involving modulus functions. However, the use of visual representations in GeoGebra provided a new dimension of graphs of modulus functions beyond participants' anticipation

Multiple representations played a pivotal role in the exploration of the graphs of modulus functions. During the first session, I noticed the slowness with which participants navigated through the screen to find correct tools or appropriate language. Once they became familiar with some features of the software, the speed of execution increased in all of Session 2. The next session focused on the comparison between the graphs of $y = |f(x)|$ and $y = f(|x|)$.

4.3 SESSION 3

Investigating and comparing the graphs of $y = |f(x)|$ and $y = f(|x|)$.

This session carried out in the third lesson extended the work on modulus functions to include $y = f(|x|)$. Session 3 focused on the comparison between the graphs of $y = |f(x)|$ and $y = f(|x|)$. Several tasks, drawn from a textbook (see Appendix A), were set to investigate graphs of $y = f(|x|)$ and $y = |f(x)|$. The session also investigated the graph of the sum of two modulus functions $y = |x + a| + |x - b|$.

The session started with a brief overview of participants' observations in the previous lessons on graphs of $y = f(x)$ and $y = |f(x)|$. Participants were involved in activities most of which concerned answering questions involving modulus functions. The first parts of the instructions recapped work from Session 1 and Session 2. We used the Interactive

Whiteboard, laptops, and some worksheets with related questions. Below each question was a grid on which participants sketched the graphs of the given function (see appendix C). A few questions that required pencil and paper sketches were given on a worksheet (see Appendix B). No grids were provided for these questions. Participants were to come up with their own sketches. The exercise was intended to establish participants' knowledge regarding the concept of modulus function graphs acquired from previous sessions. Some questions on the worksheet required participants to identify whether the graphs were for $y = |f(x)|$ or for $y = f(|x|)$. Another starter activity (see Appendix F) involved a matching task where participants matched algebraic expressions to corresponding graphical representations.

There was clear evidence from analyses of data that participants had a conceptual understanding of the relationship between the graph of $y = f(x)$ and $y = |f(x)|$, but could not always distinguish between the graph of $y = |f(x)|$ and $y = f(|x|)$.

Participants were asked to sketch the graphs of $f(x) = x + 3$, $g(x) = |x| + 3$ and $h(x) = |x + 3|$ and describe the relationship between the three functions. All six participants wrongly assumed that the graph of $g(x) = |x| + 3$ was similar to the graph of $h(x) = |x + 3|$. The initial assumption was that all graphs involving modulus functions behave in the same manner, therefore the graph of $h(x) = |x + 3|$ was perceived to be the same as that of $g(x) = |x| + 3$. Such misconception was not unique to this group of participants. A study carried out in Turkish schools (Monaghan and Ozmantar 2006) established that students became confused about the difference between ways to obtain the graph of $|f(x)|$ and $f(|x|)$.

To address misconceptions like this, de Villers (2012) recommends that teachers use the diagnostic teaching approach advocated by Bell et al (1985) and Bell (1993). In such an approach, place learners in situations which create 'cognitive conflict' between their expectations and the eventual outcome, for example, giving learners more activities like Activity (1) and Activity (2) below. Rather than viewing misconceptions as something intrinsically negative, misconceptions in this study were viewed and dealt with using de Villers' approach as an important and necessary stage of the learning process.

4.3.1 Activity 1:

Activity 1 focused on investigating graphs of $y = f(|x|)$. The expected outcome for this activity was for participants to be able to compare and describe characteristics of the graphs of $y = |f(x)|$ and $y = f(|x|)$. The long-term objective was for participants to be able to recognise and write down corresponding equations for such functions when they see them.

Instructions for this activity:

- *Use pencil and paper to sketch the graph of $y = f(|x|)$ and describe briefly how you came up with the sketch.*
- *Use GeoGebra to sketch the graph of $y = f(|x|)$ and compare it with your pencil and paper sketch*
- *Describe in your own words the relationship between the graphs of $y = f(x)$ and $y = |f(x)|$ and the graph of $y = f(x)$ and $y = f(|x|)$.*
- *Use GeoGebra to investigate graphs of modulus functions listed in the worksheet (See Appendix A).*

Vignette 11 is an extract from the dialogue that took place between me and the pair of James and Sophie as they started to investigate the graph of $y = f(|x|)$. In this vignette, I engaged in a conversation with the participants only to probe them but not making any remarks that suggested the nature of the graph of $y = f(|x|)$.

Vignette 11: Investigating the graph of $y = f(|x|)$.

.....

Line 769: James types $y = \text{abs}(x) + 3$;

Line 770: Graph of $f(x) = |x| + 3$ appears on the screen;

Line 771: James types $y = \text{abs}(x + 3)$;

Line 772: Graph of $g(x) = |x + 3|$ appears on the same screen

..... (Process of colour coding the graphs);

Line 776: Graphs appear in different colours (Fig 4.20);

Line 777: James: "Yeah, the graphs look different";

Line 778: Me: "Can you describe each graph?"

Line 779: James: "One of the graphs turns at the x-axis and the other one turns at the

y-axis”.

Line 780: Me: “Which one turns at the x-axis?”

----- (adding colour codes)

Line 791: Sophie: “The graph of $y = |x| + 3$ turns at the y-axis

Line 792: ... and the graph of $y = |x + 3|$ turns at the x-axis”

The initial pencil and paper sketch drawn by John for the graph of $y = |x| - 1$ looked the same as that for $y = |x - 1|$.

Fig 4.20: Graph of $f(x) = |x + 3|$ and $g(x) = |x| + 3$.

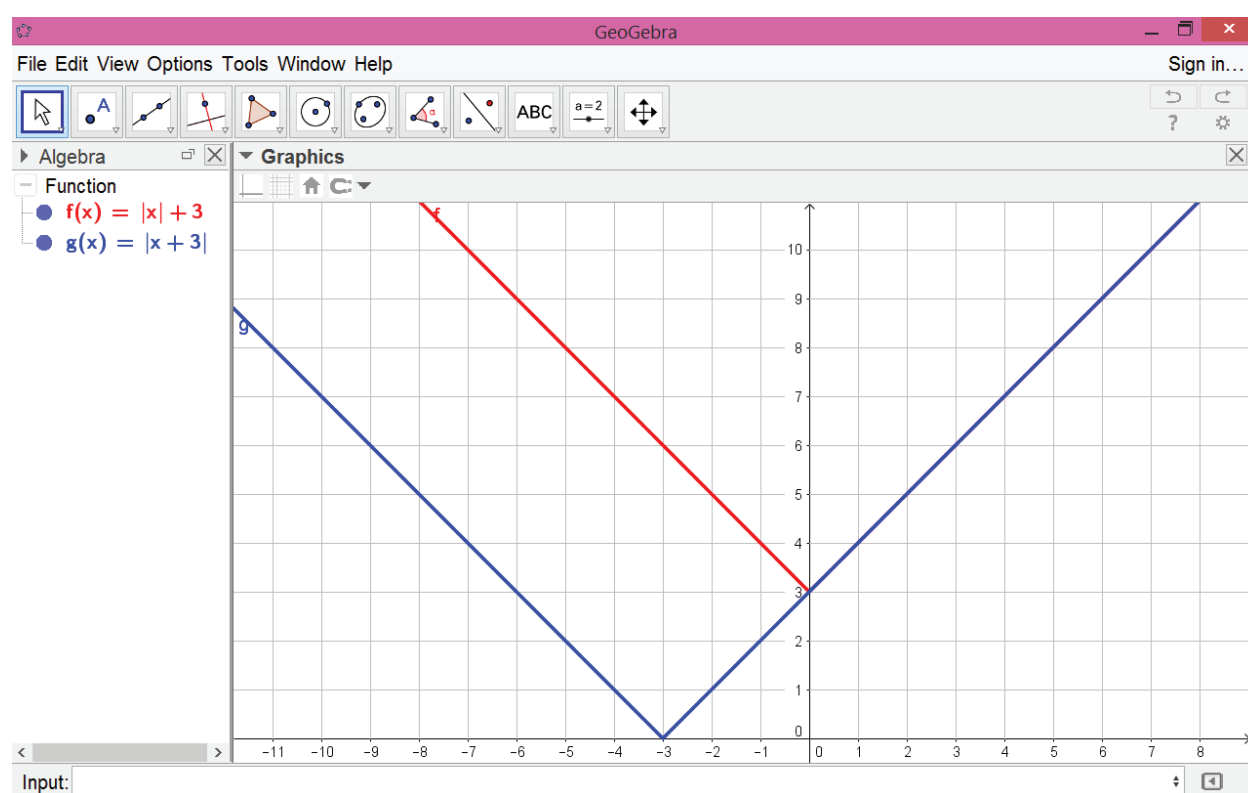


Fig 4.20 shows the graphs of $f(x) = |x + 3|$ and the graph of $g(x) = |x| + 3$. Displaying the graphs on the same axes enabled James and Sophie to compare the two and note the differences. The different diagrams helped participants to realise that the graphs of $y = |f(x)|$ and $y = f(|x|)$ are not the same, a misconception raised at the start of this session. GeoGebra displayed two different graphs as shown in Fig 4.20. From the multiple representations on the screen, James and Emma describe the behaviour in very simple terms: The graph of $y = |x| + 3$ turns at the y-axis (Line 791) and the graph

of $y = |x + 3|$ turns at the x-axis (Line 792).

4.3.2 Activity 2:

Participants were free to choose from a list of functions, which included linear, quadratic, rational, logs, exponentials, trigonometry, hyperbolic or any function of the participants' choice. In this activity, I captured the screencast clip when Susan and Peter were investigating $f(x) = x^2 - 3x$ and $g(x) = |x|^2 - 3|x|$.

Fig 4.21: Graph of $f(x) = x^2 - 3x$ and $h(x) = |x|^2 - 3|x|$.

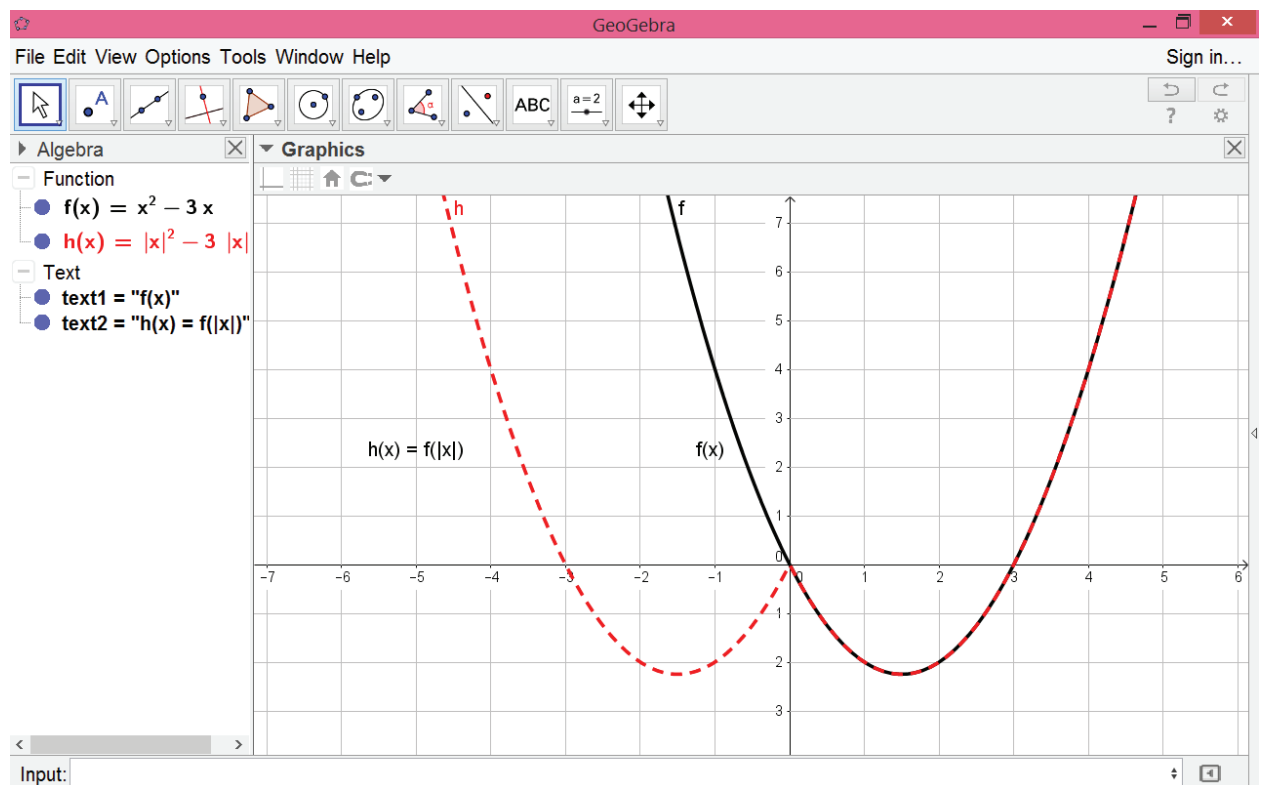


Fig 4.21 displays the graphs of $f(x) = x^2 - 3x$ and $h(x) = |x|^2 - 3|x|$. The graphs clearly show the relationship between the graph of $f(x)$ and $h(x)$. The graph of $h(x) = |x|^2 - 3|x|$ is symmetric about the y-axis. The part of the graph of $f(x)$ for $x \geq 0$ is reflected in the y-axis.

Susan and Peter were challenged to investigate more complicated graphs for polynomial such as $f(x) = x^3 + 2x^2 - 3x + 4$, and $f(|x|) = |x|^3 + 2|x|^2 - 3|x| + 4$ using GeoGebra to further establish properties of the graphs of $y = f(|x|)$.

Fig 4.22: Graphs of $f(x) = x^3 + 2x^2 - 3x + 4$, and $h(x) = |x|^3 + 2|x|^2 - 3|x| + 4$.

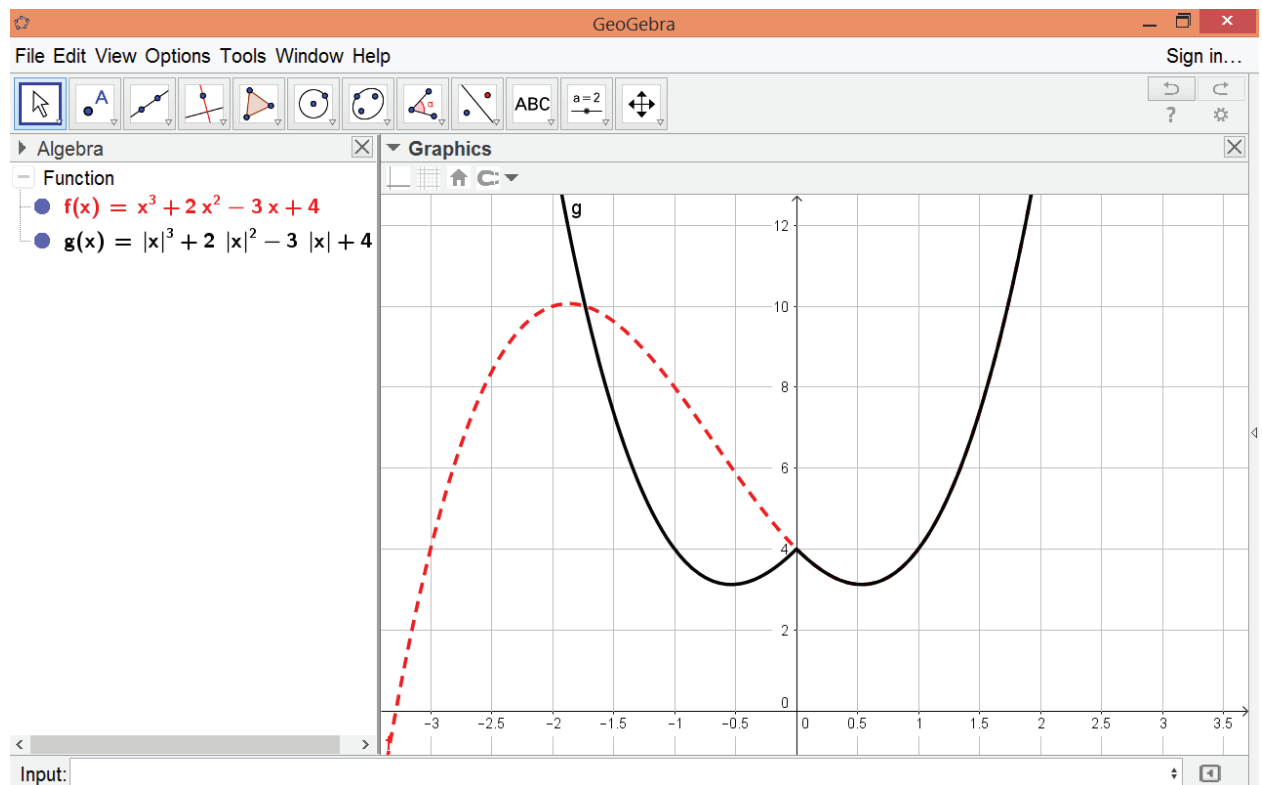


Fig 4.22 shows the graph of $f(x) = x^3 + 2x^2 - 3x + 4$, and $h(x) = |x|^3 + 2|x|^2 - 3|x| + 4$ and it further confirms observations drawn from earlier activities in this session, that the graph of $y = h(x)$ is also symmetric about the y – axis. Only the part of $f(x)$ for $x \geq 0$, is reflected in the y -axis.

Vignette 12: Investigating the graph of $y = |x|^3 + 2|x|^2 - 3|x| + 3$.

.....
Line 923: Me: “Now try to enter the function $y = |x|^3 + 2|x|^2 - 3|x| + 3$ ”;

Line 924: $y = \text{abs}(x)^3 + 2*\text{abs}(x)^2 - 3*\text{abs}(x) + 3$ is typed;

Line 925: Graphic view shows Fig 4.23;

Line 926: Algebraic view displays $f(x) = |x|^3 + 2|x|^2 - 3|x| + 3$;

Line 927: Me: “What happened to the rest of the graph?”

Line 928: Me: “Let us draw the graph of $y = x^3 + 2x^2 - 3x + 3$ on the same page”;

Line 929: $y = x^3 + 2x^2 - 3x + 3$ is typed in;

Line 930: Graphic view displays the graph in Fig 4.23;

Fig 4.23: The graphs of $f(x)$, $g(x)$ and $h(x)$.

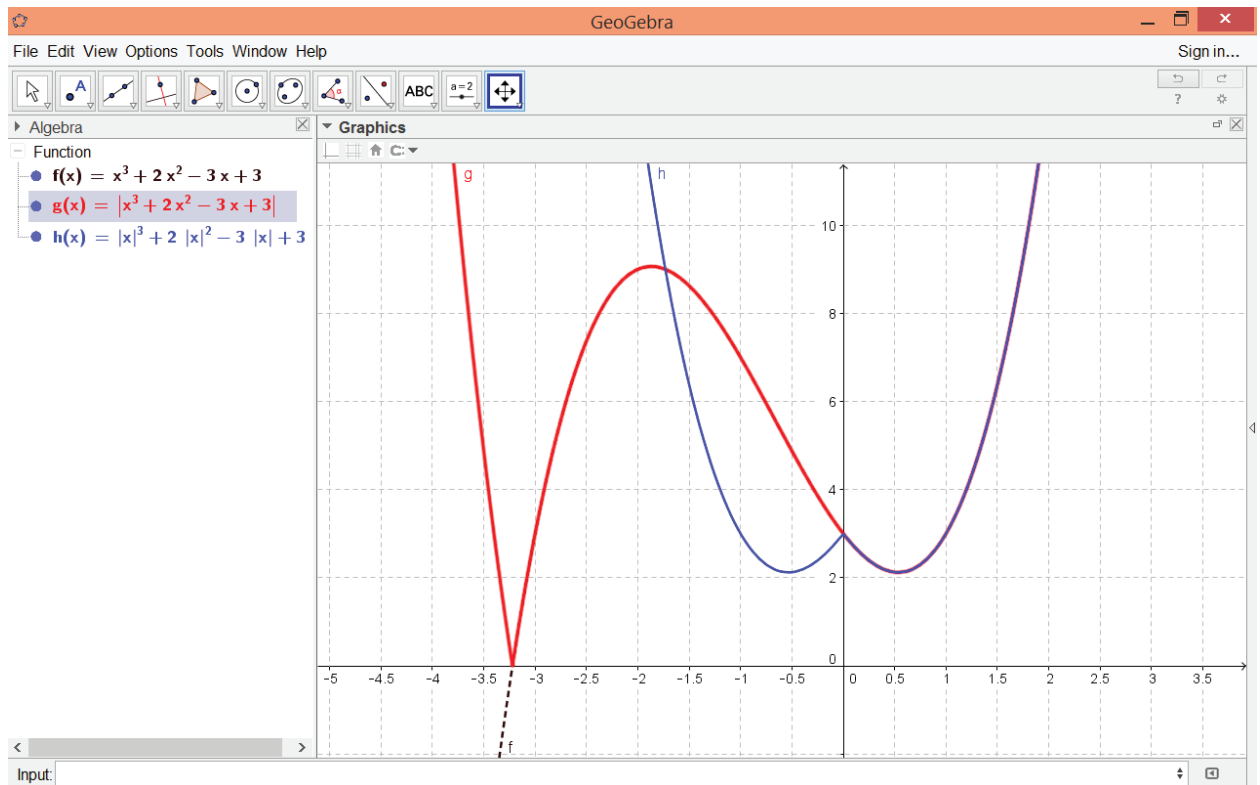


Fig 4.23 shows the graphs of $f(x) = x^3 + 2x^2 - 3x + 3$, $g(x) = |x^3 + 2x^2 - 3x + 3|$ and $h(x) = |x|^3 + 2|x|^2 - 3|x| + 3$ on the same plane.

Fig 4.24: The graphs of $f(x) = e^x$ and $g(x) = e^{|x|}$.

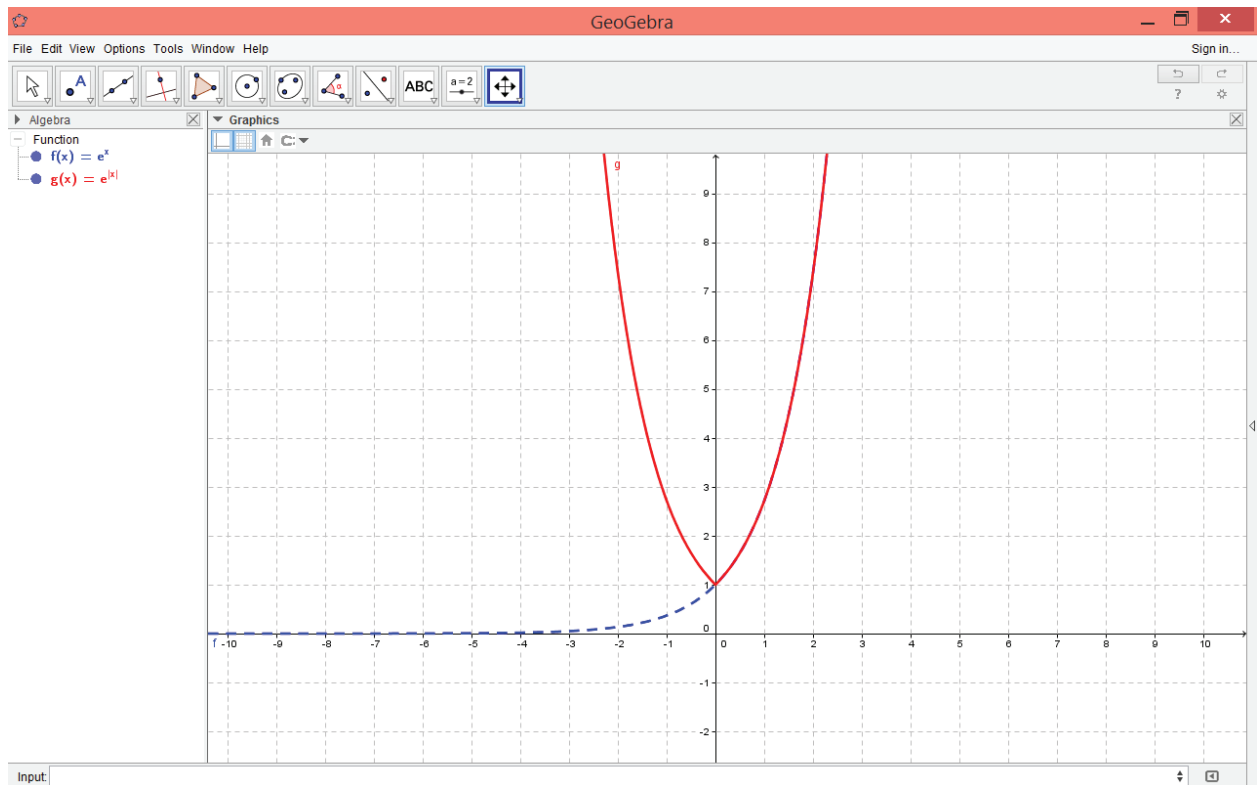


Fig 4.24 shows the graph of $f(x) = e^x$ and $g(x) = e^{|x|}$. Vignette 13 is an extract of the conversation between Emma and John while they were investigating the graphs $f(x) = e^x$ and $g(x) = e^{|x|}$.

Vignette 13: Investigating graphs of $f(x) = e^x$ and $g(x) = e^{|x|}$.

Line 824: The equation $y = e^x$ is entered and another graph (blue) pops up

Line 825: Another equation $y = e^{(abs(x))}$ is entered

.....

Line 834: John: “The graph of $g(x) = e^{|x|}$ is a reflection of the graph of $f(x) = e^x$ in the y – axis”.

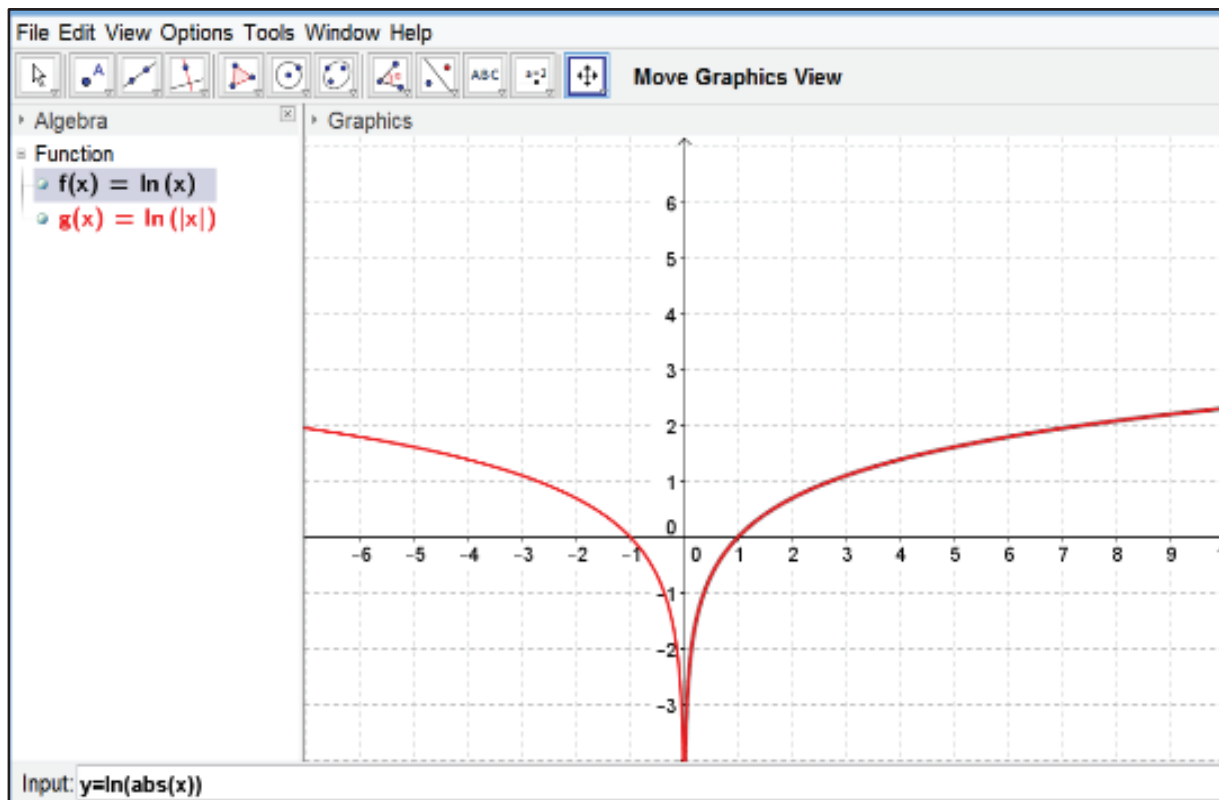
Line 835: Emma: “It is only that part of the graph that is on the right of the y -axis that is reflected in the y -axis”.

Line 836: John: “Hang on! The graph of $y = |f(x)|$ is a reflection of $y = f(x)$ in the x -axis and $y = f(|x|)$ is a reflection in the y -axis”.

Line 834: Emma: “The graph of $y = f(|x|)$ and $y = |f(x)|$ are different”.

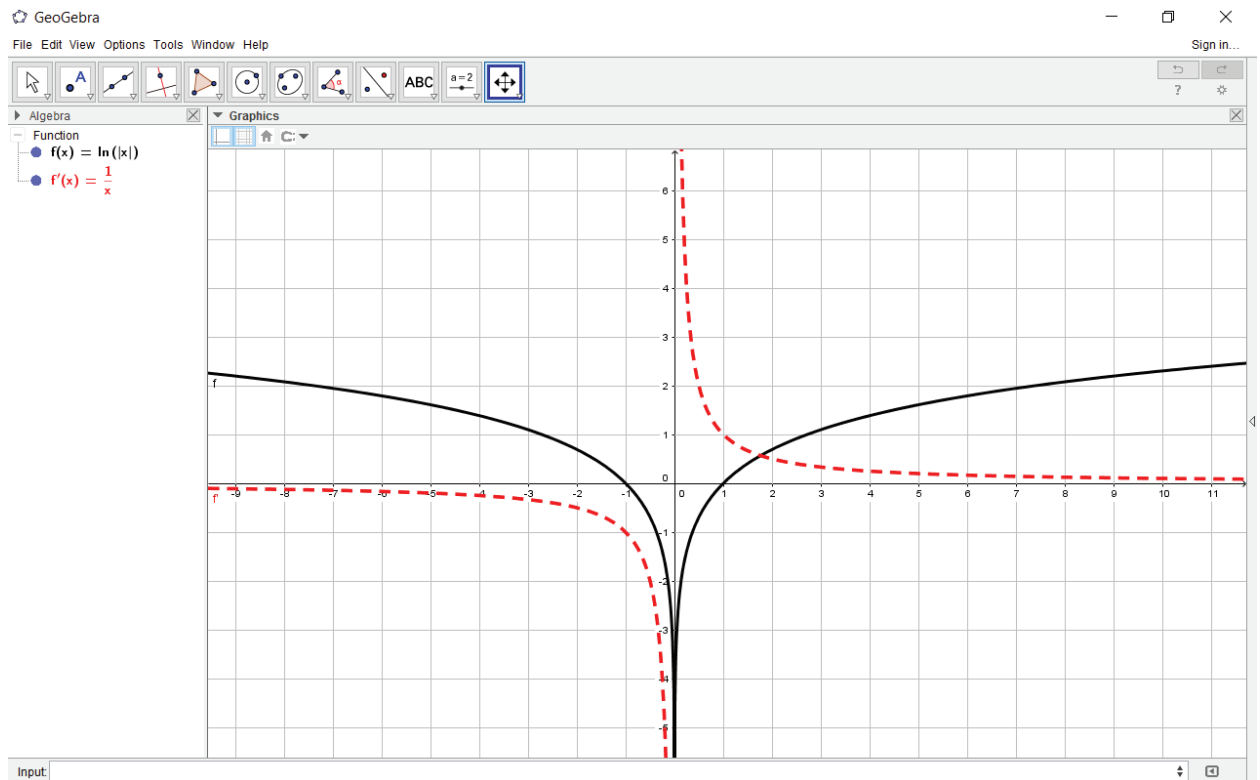
Emma and John concluded that the graph of the modulus function $y = e^{|x|}$ was a reflection of the part of the graph of $y = e^x$ for $x > 0$ in the y-axis. The occurrence of the same characteristics from the generated functions led to generalisations of the characteristics of the graphs of $y = f(|x|)$. In Line 834, Emma noted the differences between the graph of $y = f(|x|)$ and $y = |f(x)|$

Fig 4.25: Graphs of $y = \ln(x)$ and $y = \ln(|x|)$.



The investigation of the graph of $f(x) = \ln(x)$ and $g(x) = \ln(|x|)$ on the same axes as shown in Fig 4.25 raised some discussion points. A glance at the graphs in Fig 4.25 further confirmed the accession that the graph of $f(x) = \ln(x)$ is all reflected in the y-axis. However, James raised a very interesting observation (see Vignette 14).

Fig 4.26: The graph of $f(x) = \ln(|x|)$ and its gradient function $f'(x)$ displayed in GeoGebra.



Vignette 14 is an extract of conversation captured from participants discussions provoked by the visual displays shown on GeoGebra, which contradicted participants' understanding of derivative of $\ln(x)$ and the integration of $1/x$.

Vignette 14, Similarities between the graphs of the derivative functions of $\ln(x)$ and $\ln|x|$.

James: "When we used GeoGebra to differentiate $f(x) = \ln(x)$ we got a rectangular hyperbola for $f'(x) = \frac{1}{x}$ (Fig 4.15). We are also getting the same rectangular hyperbola for the derivative of $\ln|x|$. Why is the $\int \frac{1}{x} dx$ given as $\ln|x|$ only? The graph of $y = \ln(x)$ is different from that of $y = \ln|x|$ but the graphs of their gradient functions are the same.

Peter: "Your observation is not entirely wrong. The problem is that the domain of $\ln(x)$ is for $x > 0$, while $\frac{1}{x}$ is defined everywhere except at $x = 0$. The function that matches that domain is $\ln|x|$.

Sophie: “Peter has a point. It has to do with area under the curve. We can definitely see from the graph that $\frac{1}{x}$ is defined everywhere except when $x = 0$ and we can find $\int_{-2}^{-1} \frac{1}{x} dx$. If we do not use the absolute value, we will not be able to find natural log of negative numbers, even though we know that this area exists”.

James: “Why not just say $\frac{1}{x}$ is the derivative of $y = \ln|x|$ so that the integral of $\frac{1}{x}$ gives back $\ln|x|$ ”.

Sophie: “For $x > 0$, you are correct. The derivative of $\ln|x|$ is the same as the derivative of $\ln(x)$. But the graph of $\ln|x|$ and its derivative are both defined for values of x less than zero as well”.

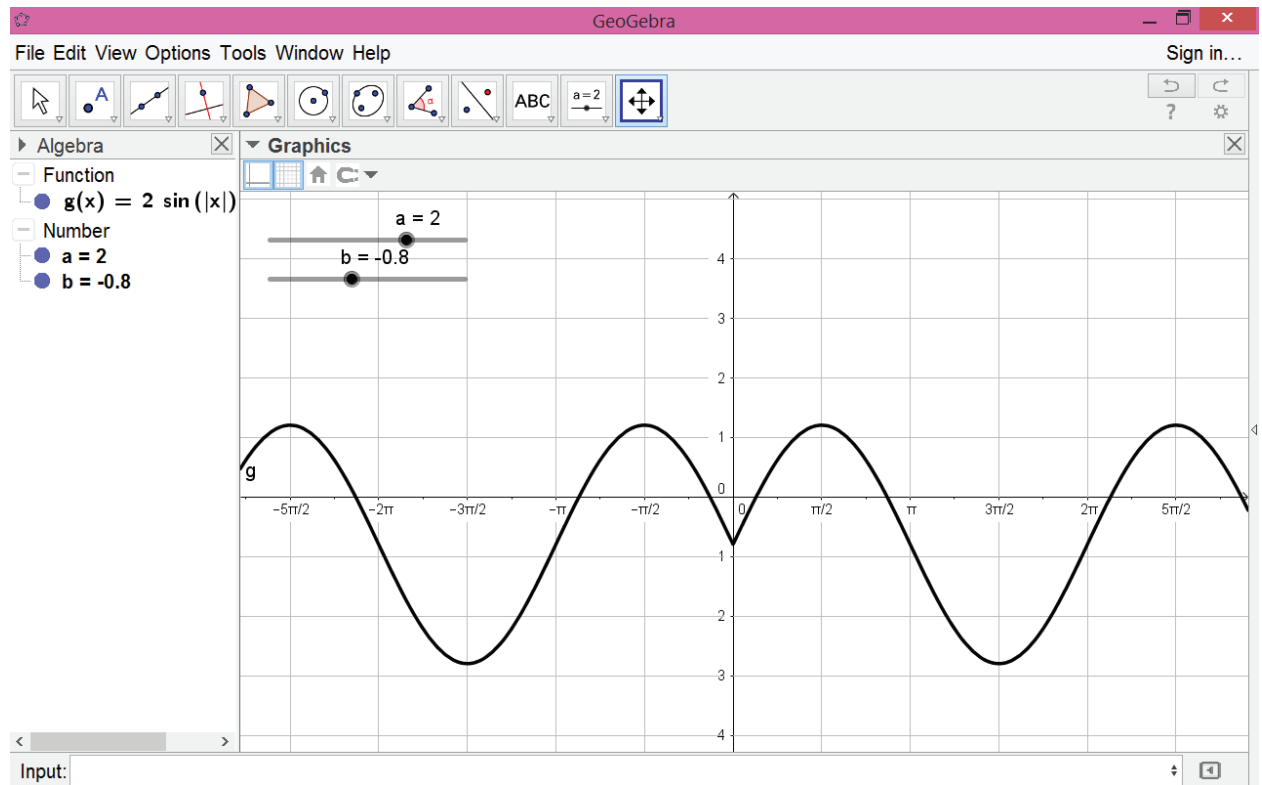
John: “For me, I take it that integration serves two purposes: to solve differential equations or to find area under the curve. If I am given $\frac{dy}{dx} = \frac{1}{x}$ and I am required to find y , I will give it as $y = \ln(x) + c$. But if I am asked to $\int_a^b \frac{1}{x} dx$, the rectangular hyperbola exists on both sides of the y – axis. I will give my answer as $\ln|x|$. This will enable me find area on either side of the y -axis.

Susan: Thank you Sophie. I really never realised that integration served different purposes.

The discussion in Vignette 14 involved all participants. The different representations for $f(x) = \ln(x)$ (Fig 4.26) and $g(x) = \ln|x|$ (Fig 4.25) evoked a discussion that initially centred on the correctness of the representations displayed in GeoGebra. James questioned why the integral of $1/x$ was not just given as $y = \ln(x)$. Each participant presented an argument to justify the correctness of the representations. John highlighted that integration served two purposes. Sophie talked about the domains of the functions and justified why the integral of $\frac{1}{x}$ is $\ln|x|$. The multiple representations of the GeoGebra screen generated discussions and allowed participants to focus on issues of understanding the differences between the functions $\ln(x)$ and $\ln|x|$ including their derivatives.

This section is not a logical follow up to the discussions highlighted above, but screenshot clips taken from other recorded video.

Fig 4.27: The graph of $g(x) = a \sin |x| + b$.



In all examples considered, there was an element of the graph of $y = f(|x|)$ being symmetric about the $y =$ axis. It was generally agreed that, to sketch the graph of $y = f(|x|)$, first sketch the graph of $y = f(x)$. Then take any part of the graph of $y = f(x)$ for all values of $x \geq 0$ and reflect it in the y -axis. The y -axis remains a permanent line of symmetry for all graphs involving $f(|x|)$ with or without any transformations involved.

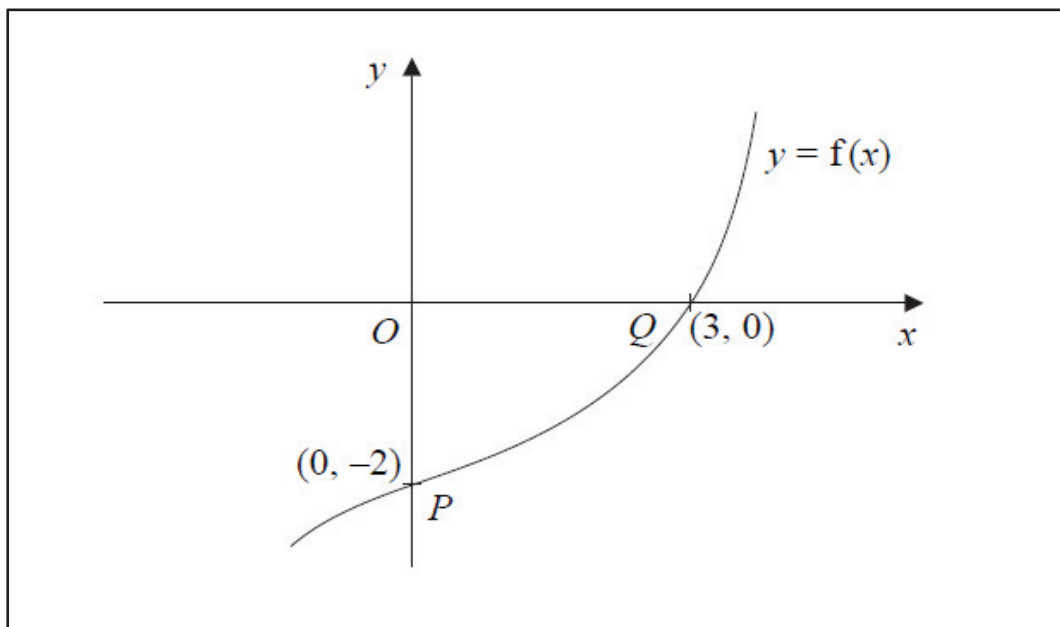
Working in a GeoGebra environment not only gave participants accurate diagrams but also allowed students to experiment more while appreciating the ease of getting many examples (Yerushalmy 1997). Repeating the same process over and over again helped to confirm the relationship between the graph of $y = f(x)$, $y = |f(x)|$ and $y = f(|x|)$. Participants explored as many cases as time could permit, without even considering the complexity of some of the questions. The focus shifted from sketching correct graphs using pencil and paper to investigating properties of the graphs of modulus functions.

Information obtained through use of multiple representations was a step towards generation of conjectures, which in turn served as a basis for learning the concept of modulus functions and generalising observations.

4.3.3 Activity 3: Consolidating of work:

In this activity, participants worked through questions selected from past examination questions. No GeoGebra was used in this activity. The focus of the activity was to check participants' understanding of the relationship between the graphs of $y = f(x)$, $y = |f(x)|$ and $y = f(|x|)$. The questions displayed sketches of $y = f(x)$. Participants used pencil and paper to sketch the respective graphs of $y = |f(x)|$ and $y = f(|x|)$ for each given function.

Fig 4.28: Question 1:



[Source: Adopted from Edexcel C3: June2006 Qu. 3]

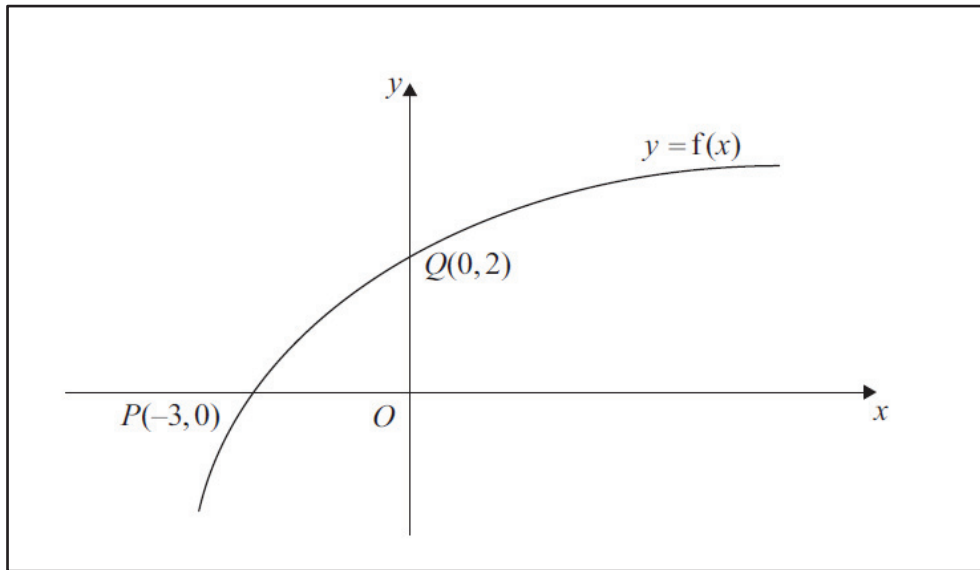
The diagram shows part of the curve with equation $y = f(x)$, $x \in \mathbb{R}$, where f is an increasing function of x . The curve passes through the points $P(0, -2)$ and $Q(3, 0)$ as shown. On a separate diagram, sketch the curves with equations;

- (a) $y = |f(x)|$
- (b) $y = f(|x|)$

All the six participants seemed more confident and clear about the graphs of $y = |f(x)|$ and $y = f(|x|)$. They correctly identified the images of $P'(0, 2)$ and $Q'(-3, 0)$ for the graph of $y = |f(x)|$. However, four out of the six correctly sketched the graph of $y = |f(x)|$. Of the

two who failed, one had a sketch that appeared correct, but with no coordinates for Q' . Susan failed to sketch the graph of $y = f(|x|)$ completely. She sketched a graph similar to that of $y = |f(x)|$. This is not consistent with the rest of her work in other questions. She attributed this to a careless mistake.

Fig 4.29: Question 2:

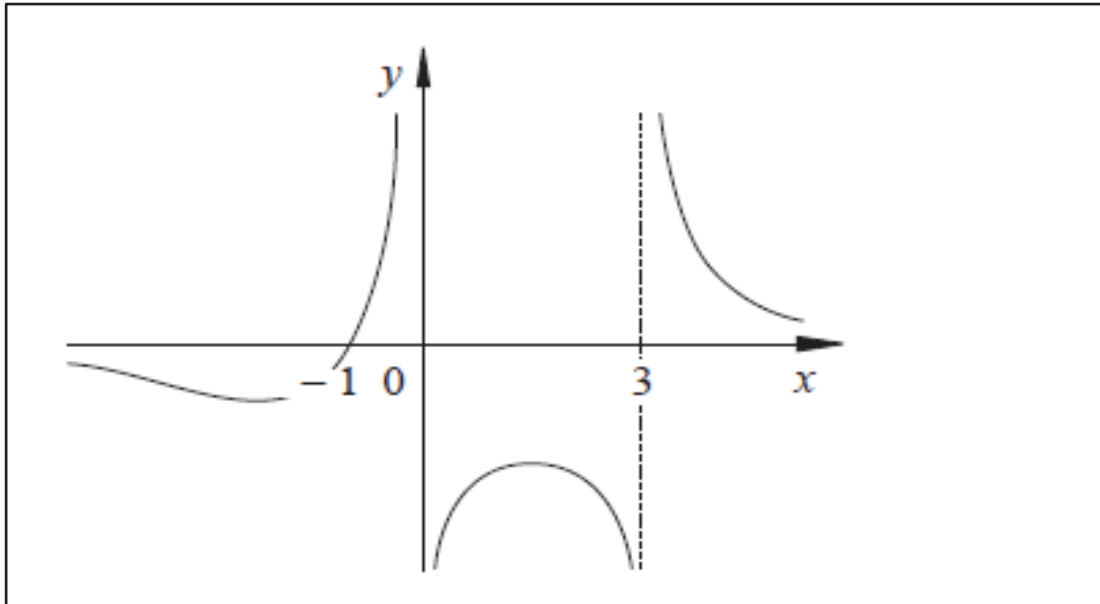


The diagram shows part of the curve with equations $y = f(x)$, $x \in \mathbb{R}$. The curve passes through the points $Q(0, 2)$ and $P(-3, 0)$ as shown. On separate diagrams, sketch the curve with equation $y = f(|x|) - 2$, (Edexcel C3: Jan 2013 Qu. 3).

There is evidence (Appendix D) of participants' understanding of modulus functions, but the inclusion of a transformation confused all but one. Only John managed to sketch the graph of $y = f(|x|) - 2$ correctly indicating the coordinates of Q' (0, 0) correctly. The others struggled with this question. They gave coordinates of P as (-3, -2). They did not realise that P disappears under this transformation because it is on the left side of the y-axis. Participants failed to transfer knowledge from work covered in the activities.

I created Question 3 from the graph of $y = \frac{x+1}{x(x-3)}$ to challenge participants' understanding of modulus functions.

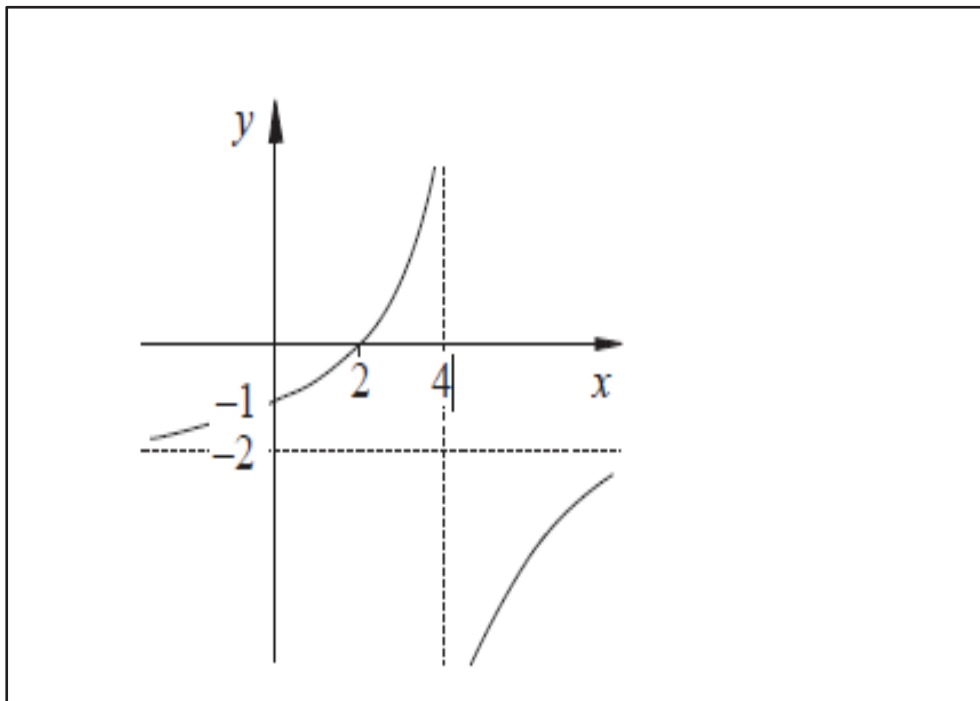
Fig 4.30: Question 3:



The diagram shows the graph of $y = f(x)$. On separate diagrams, sketch the graphs of $y = |f(x)|$ and $y = f(|x|)$.

All the six participants managed to sketch the graphs of $y = |f(x)|$ and $y = f(|x|)$ correctly (see Appendix D). There was no mix-up between the two graphs. The sketches do not show enough evidence that participants would have got the correct answers if coordinates had been included in this question.

Fig 4.31: Question 4: Created from the graph of $y = \frac{2x-7}{4-x}$



The diagram shows the graph of $y = f(x)$. On separate diagrams, sketch the graph of $y = |f(x)|$ and the graph of $y = f(|x|)$.

The responses to this question were mixed. Emma, Peter and Sophie failed to reflect the part of the graph below $y = -2$ (for $x > 4$) in the x-axis. The other three participants correctly sketched the graph of $y = |f(x)|$. Only James managed to correctly sketch the graph of $y = f(|x|)$. The rest failed to reflect the asymptote $x = 4$ in the y-axis, hence their image is shown immediately after the y-axis (see Appendix D).

Questions 1 – 4 presented graphs without functional equations. Monaghan and Ozmantar (2006) argue that presenting graphs without functional equations aid students to develop a rule to obtain the intended graphs by analysing the graphs alone. In this exercise, there was no opportunity for participants to use GeoGebra to sketch the required graphs.

Question 5

The function f is defined by $f: x \rightarrow \ln(2x - 1)$ for $x > \frac{1}{2}$ and g is defined by $g: x \rightarrow \frac{2}{x-3}$ for $x \in \mathbb{R}$ and $x \neq 3$. Sketch the graph of $y = |g(x)|$. Indicate clearly the equation of the vertical asymptote and the coordinates of the points where the graph crosses the y -axis.

- a) On the same grid sketch the graph of $y = f(x)$.
- b) Estimate the value of $\left| \frac{2}{x-3} \right| = \ln(2x-1)$. (Source: Adopted from Edexcel C3: June2007, Qu. 5)

Question 5 was given without the aid of the diagram and participants were free to use GeoGebra. Sketching the graph of the rational function $\frac{2}{x-3}$ posed some problems for all participants. However with the background knowledge of transformation of the $y = a f(x + b)$ it was easy to guide participants through sketching the graph of $\frac{2}{x-3}$ from the graph of $1/x$. Susan, Sophie and James used GeoGebra to check their solutions. All participants managed to sketch the graphs of $y = f(x)$ and $y = |g(x)|$ correctly, which was a reflection of their understanding of graphs of modulus functions. Solving the equation graphically did not yield accurate answers since their sketches were not very accurate. GeoGebra was used to find correct solutions.

4.3.3.1 Participants' written responses:

Data obtained from written responses (see Appendix D) suggests that participants made use of the knowledge obtained from the use of GeoGebra to correctly sketch some graphs of $y = |f(x)|$ and $y = f(|x|)$ but failed in other instances where graphs looked complicated. Though the activity displayed questions without the functional equations, participants developed correct perceptions to obtain the intended graphs. Participants used knowledge from earlier activities such as:

- i. graph of $y = |f(x)|$ is a reflection of the positive part of the graph of $y = f(x)$ in the x – axis,
- ii. the graph of $y = f(|x|)$ is a reflection of the part of the graph of $y = f(x)$ on the right of the y -axis in the y -axis. They correctly drew graphs of $y = |f(x)|$ and $y = f(|x|)$.

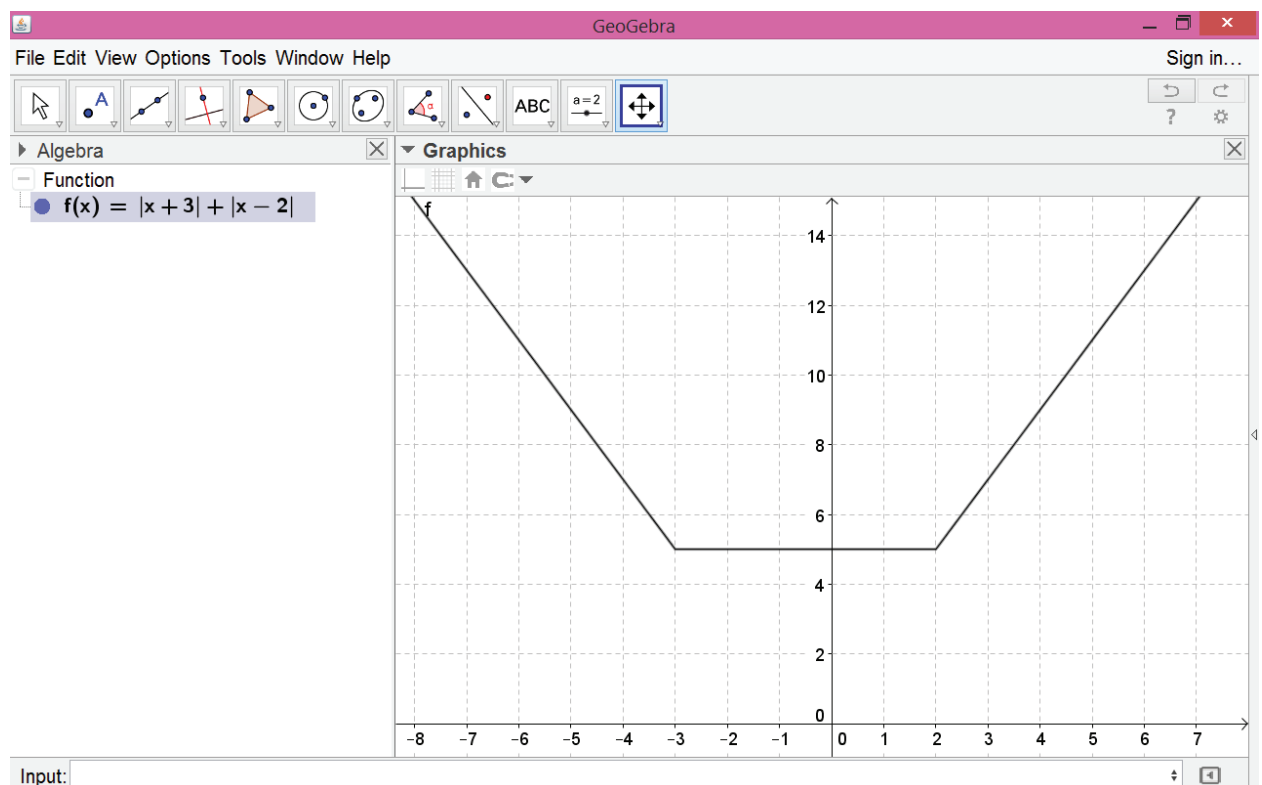
Session 3 Activity 4:

Investigating graphical representations for the sum of two modulus functions $|x+a| + |x-b|$:

As a stretching and challenging activity, but within the confines of the curriculum, participants investigated the graph of $y = |x + a| + |x - b|$. The general question posed at the beginning of the activity was: “If $f(x)$ is a sum of two linear modulus functions $g(x)$ and $h(x)$, can you draw the graph of $|g(x)| + |h(x)|$ from the graph of $f(x) = g(x) + h(x)$?”

John looked at the function $f(x) = |x + 3| + |x - 2|$, ignored the modulus functions and added the right hand side, wrongly yielding a linear function $f(x) = |2x + 1|$. He then sketched the graph of $f(x) = |2x + 1|$. I challenged him to input $f(x) = |x + 3| + |x - 2|$ in GeoGebra. He entered the function $f(x) = \text{abs}(x + 3) + \text{abs}(x - 2)$. Fig 4.32 appeared on the screen.

Fig 4.32: The graph of $f(x) = |x + 3| + |x - 2|$ as displayed in GeoGebra.



From his reaction, John did not expect to see what came up on the screen. All participants confessed that they had not anticipated a representation similar to what

appeared on John's screen. John summarised his interpretation of the graphical representations.

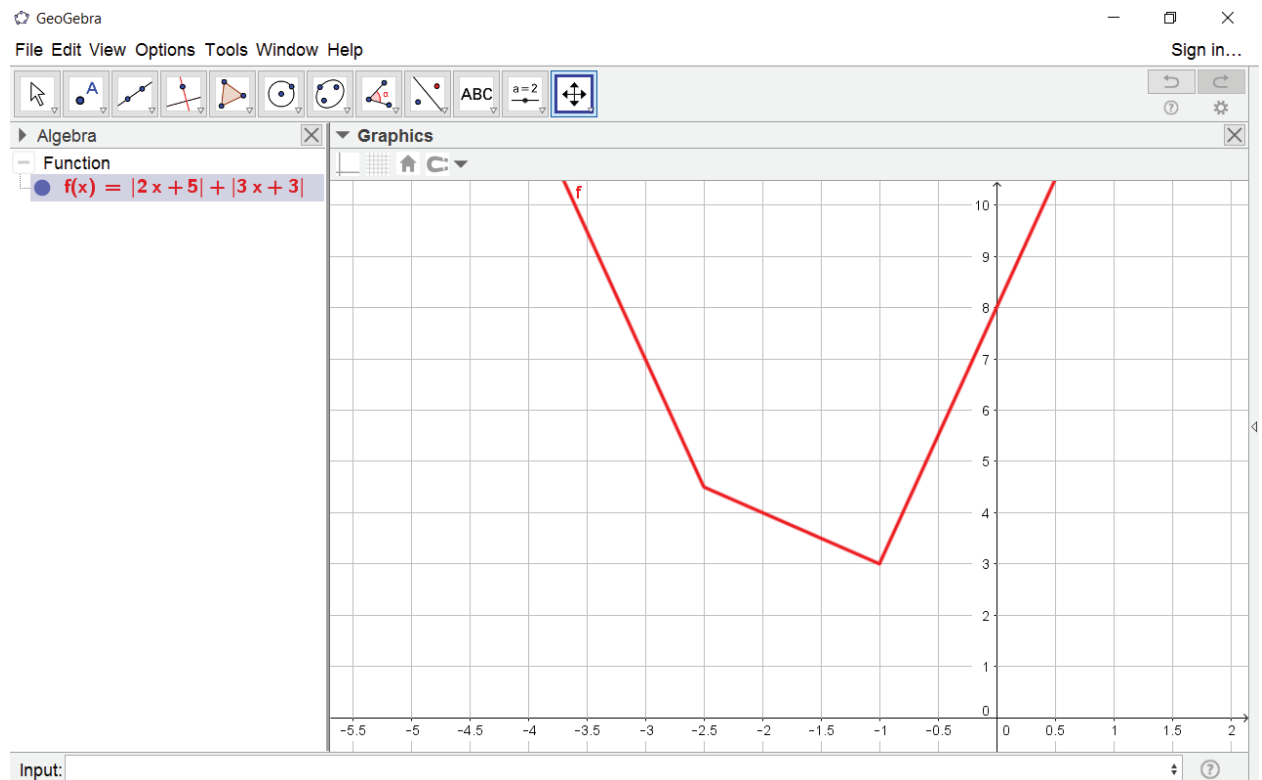
- i. $f(x) = x + 3$ for $x \geq 2$
- ii. $f(x) = 3 + |-2|$ for $-3 \leq x \leq 2$
- iii. $f(x) = x - 2$ for $x \leq -2$
- iv. The length of the horizontal line is the distance between the parameters -3 and 2.

This raised some discussion points and participants were encouraged to consider the algebraic manipulation of the sum of the modulus functions. Susan led the discussions:

- If $|x + 3| = x + 3$ and $|x - 2| = x - 2$, then $x + 3 + x - 2 = 2x + 1$ for $x > 2$
- If $|x + 3| = x + 3$ and $|x - 2| = -x + 2$, then $x + 3 - x + 2 = 5$ for $-3 < x < 2$
- If $|x + 3| = -x - 3$ and $|x - 2| = -x + 2$, then $-x - 3 - x + 2 = -2x - 1$ for $x < -3$

John argued that from the graph (see Fig 4.32) the third part should be $-2x + 1$. His comparison of the graphical representation and algebraic representation did not give him the clarity he was looking for. He however acknowledged that his earlier assumptions were incorrect.

Fig 4.33: Graph of two modulus functions $f(x) = |2x + 5| + |3x + 3|$.



Further investigations using sliders for $f(x) = |2x + 5| + |3x + 3|$ (see Fig 4.33) led to the general conclusion that the graph of $f(x)$ is divided into 3 sections

- i) $f(x) = 5x + 8$ for $x > -1$
- ii) $f(x) = -x + 2$ for $-5/2 < x < -3/3$ (i.e. $-2.5 < x < -1$)
- iii) $f(x) = -5x - 8$ for $x < -2.5$

There was no attempt from the participants to give general descriptions for the graphs of the sum of two modulus functions $f(x) = |ax + b| + |cx + d|$, but Peter noted that the middle section of the graph is horizontal if $a = c$.

The next session looked at equations involving modulus functions. Participants had a choice to work the solutions algebraically or use the aid of GeoGebra to get the visual representations before attempting the questions algebraically.

SESSION 4:

Solving equations involving modulus functions:

The fourth session focused on solving equations involving modulus functions, but also left room for exploring graphs of more complicated modulus functions. At all times during this session, participants had access to the laptops and GeoGebra.

Session 4 Activity 1:

Participants got a worksheet with a set of equations involving modulus functions. Participants produced a variety of solutions, but for this study, I have selected cases that provide rich data for my research questions and some cases where participants made use of GeoGebra in the process of solving the equations.

Question: Find the value(s) of x for which $|x - 2| = |4x - 3|$.

Insert 4.2 is an extract from Peter's hand written work.

Insert 4.2

$$\begin{aligned} \Rightarrow |x-2| &= x-2 \text{ and } |4x-3| = 4x-3 \\ x-2 &= 4x-3 \\ +1 &= 3x \\ x &= \frac{1}{3} \text{ and } y = -\frac{5}{3} \\ \text{Coordinates } &(\frac{1}{3}, -\frac{5}{3}) \\ \Rightarrow |x-2| &= x-2 \text{ and } |4x-3| = -4x+3 \\ x-2 &= -4x+3 \\ 5x &= 5 \\ x &= 1 \text{ and } y = -1 \\ \text{Coordinates } &(1, -1) \\ \Rightarrow |x-2| &= -x+2 \text{ and } |4x-3| = 4x-3 \\ -x+2 &= 4x-3 \\ 5 &= 5x \\ x &= 1 \quad y = -1 \\ \text{Coordinates } &(1, -1) \\ \Rightarrow |x-2| &= -x+2 \text{ and } |4x-3| = -4x+3 \\ -x+2 &= -4x+3 \\ 3x &= 1 \\ x &= \frac{1}{3} \quad y = \frac{5}{3} \\ \text{Coordinates } &(\frac{1}{3}, \frac{5}{3}) \end{aligned}$$

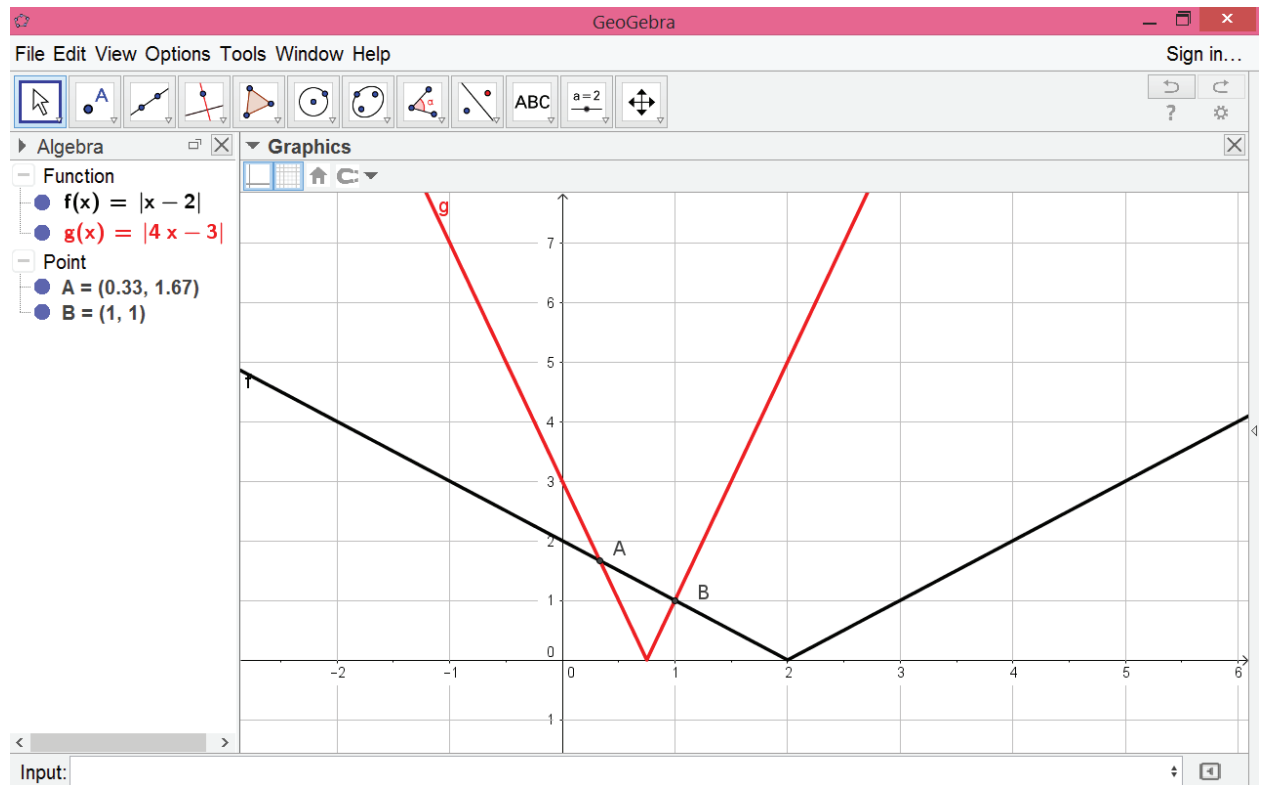
Peter considered case by case to identify all possible values of x for which the equation was true. He gave four coordinates $(1/3, -4/3)$, $(1, -1)$, $(1, 1)$ and $(1/3, 4/3)$ in his answer, but failed to note that $|x - 2|$ or $|4x - 3|$ are always positive, hence the y -coordinate cannot be negative. GeoGebra correctly displayed two points A $(1/3, 4/3)$ and B $(1, 1)$ as the only solutions to the equation $|x - 2| = |4x - 3|$ (see Fig 4.34). Points $(1/3, -5/3)$ and $(1, -1)$, below the x -axis were outside the range of the two given modulus functions. The graphs did not exist below the x -axis.

With the aid of a diagram, Peter realised that he could have limited his investigations to Case 3 and 4 (see Fig 4.34) only and discarded Cases 1 and 2. His execution of the algebraic calculations was correct, but yielded more solutions than required. Besides producing spurious answers, the algebraic method was long and tedious. Sketching the graphs of $f(x) = |x - 2|$ and $g(x) = |4x - 3|$ first, would limit the number of cases to be investigated to Cases 3 and 4 only.

The algebraic method of solving equations involving modulus functions sometimes produces spurious answers as seen from Peter's work. GeoGebra provided visual representations, which later helped participants to restrict working to focus on what was necessary. After analysing Peter's work as a group, we concluded that it is advisable to sketch the graph before solving the equations algebraically. A correct sketch would show the number of expected solutions before applying the algebraic method.

Peter's approach of using an algebraic method tallies with Yerushalmy (1991) who found that even after extensive experience with multi-representational learning experiences designed to teach understanding of functions, only twelve per cent of students gave answers that involved both numerical and visual representations. Most answers reflect the use of one representation and a neglect of the other. Yerushalmy's research suggests that appreciating the links across multiple representations is not automatic. Poppe (1993) explored the effects of differing technological approaches to calculus on students' use and understanding of multiple representations when solving problems. Poppe found that although students realised that tables, graphs and mapping diagrams were helpful, they did not use them in order to solve unfamiliar mathematical problems unless it was suggested to them.

Fig 4.34: Graphs of $f(x)$ and $g(x)$ displayed on the same screen.



This diagram (Fig 4.34) clearly shows that there are only two points of intersection at A (1/3, 4/3) and B (1, 1), hence two solutions of the equation $|x - 2| = |4x - 3|$. On the Interactive whiteboard, I used a similar equation to demonstrate to the group why Peter got four points instead of just two (See Fig 4.35).

Fig 4.35: Solving $|2x + 1| = |x + 2|$ graphically.

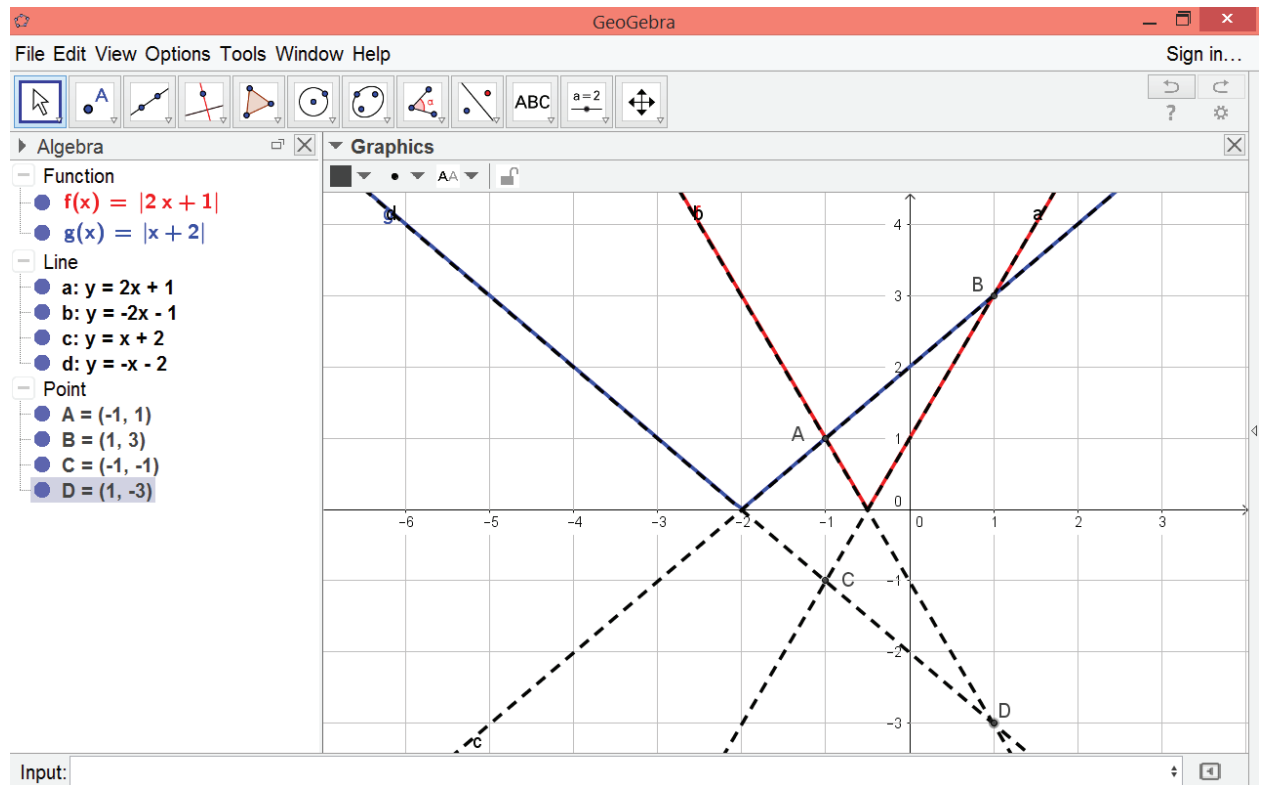
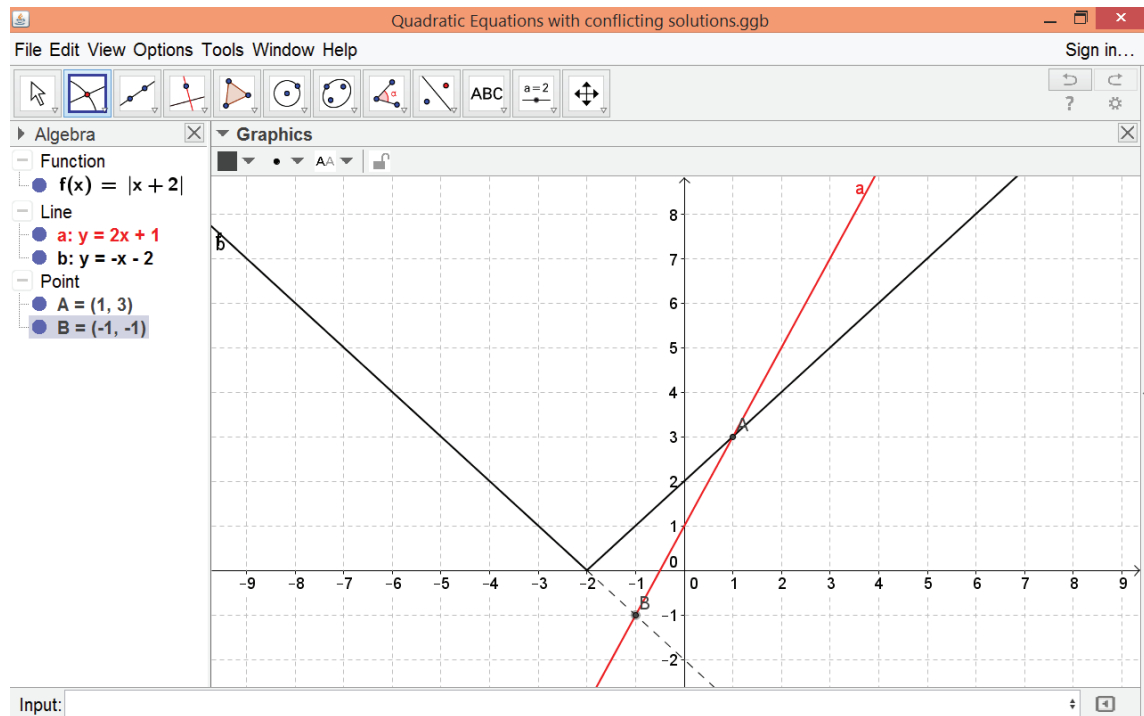


Fig 4.35 illustrates that if the domain of the graphs of modulus functions is not restricted, there are exactly 4 points of intersection A $(-1, 1)$, B $(1, 3)$, C $(-1, -1)$ and D $(1, -3)$.

Fig 4.36: Solving $|x + 2| = 2x + 1$ graphically.



Question: Find possible values of x for which $|x + 2| = 2x + 1$

Emma's written response is shown in Insert 4.3:

Insert 4.3

$$\begin{aligned}
 |x+2| &= 2x+1 \\
 \therefore x+2 &= 2x+1 \\
 x &= 1 \\
 \text{OR } |x+2| &= 2x+1 \\
 -x-2 &= 2x+1 \\
 -3x &= 3 \\
 x &= -1
 \end{aligned}$$

Emma discarded the second solution after illustrating the two functions on the GeoGebra screen. Fig 4.36 clearly illustrates that the only lines that intersect are $y=2x+1$ and $y = x + 2$. The intersection of $y = 2x + 1$ and $y = -x - 2$ falls outside the range of the modulus function graphs discarding point $(-1, -1)$. The graphical representation clearly shows that the only valid solution for this equation is $x = 1$, which occurs at point A $(1, 3)$.

Participants solving equations involving modulus functions algebraically:

Although participants used several approaches to solve modulus function equations, they continued to use GeoGebra to check their solutions. Below are some extracts from participants' algebraic work before using GeoGebra to check.

Question 1: Solve the equation $|x| = |2x + 1|$

Susan squared both sides

Insert 4.4

$$\begin{aligned} |x| &= |2x + 1| \\ x^2 &= (2x + 1)^2 \\ x^2 &= 4x^2 + 4x + 1 \\ 3x^2 + 4x + 1 &= 0 \\ (3x + 1)(x + 1) &= 0 \\ x &= -\frac{1}{3} \text{ or } x = -1 \end{aligned}$$

Question 2: Solve the equation $|x - 1| = -3$.

In this question Susan continues with the same method but fails to realise that the method of squaring both sides only works if the function or number on the right side is positive.

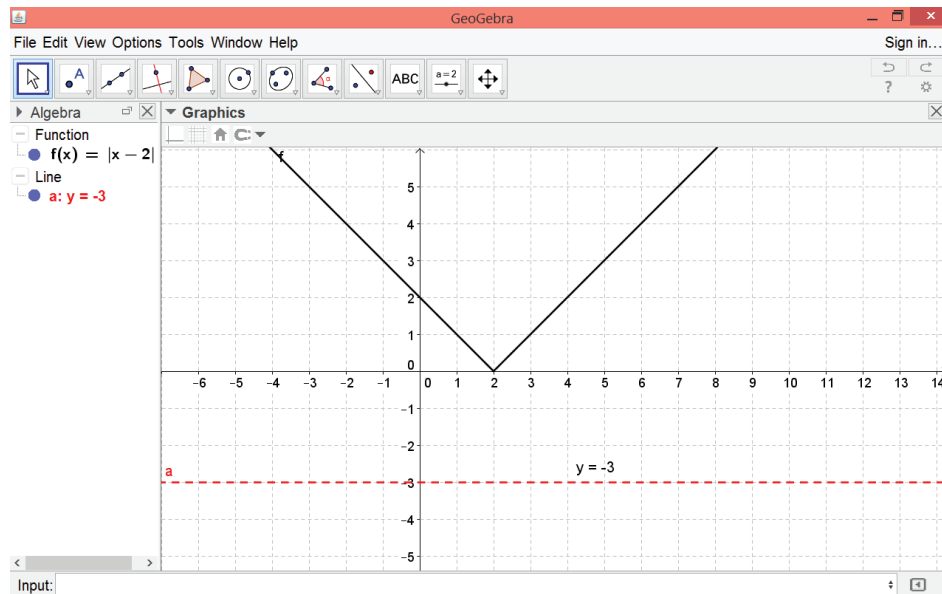
Insert 4.5

$$\begin{aligned} |x - 3| &= -3 \\ (x - 3)^2 &= (-3)^2 \\ x^2 - 6x + 9 &= 9 \\ x^2 - 6x &= 0 \\ x(x - 6) &= 0 \\ x &= 0 \text{ and } x = 6 \end{aligned}$$

When the two functions $f(x) = |x - 2|$ and $y = -3$ were entered in GeoGebra (see Fig 4.37), the graphical representation clearly showed that the two graphs did not intersect,

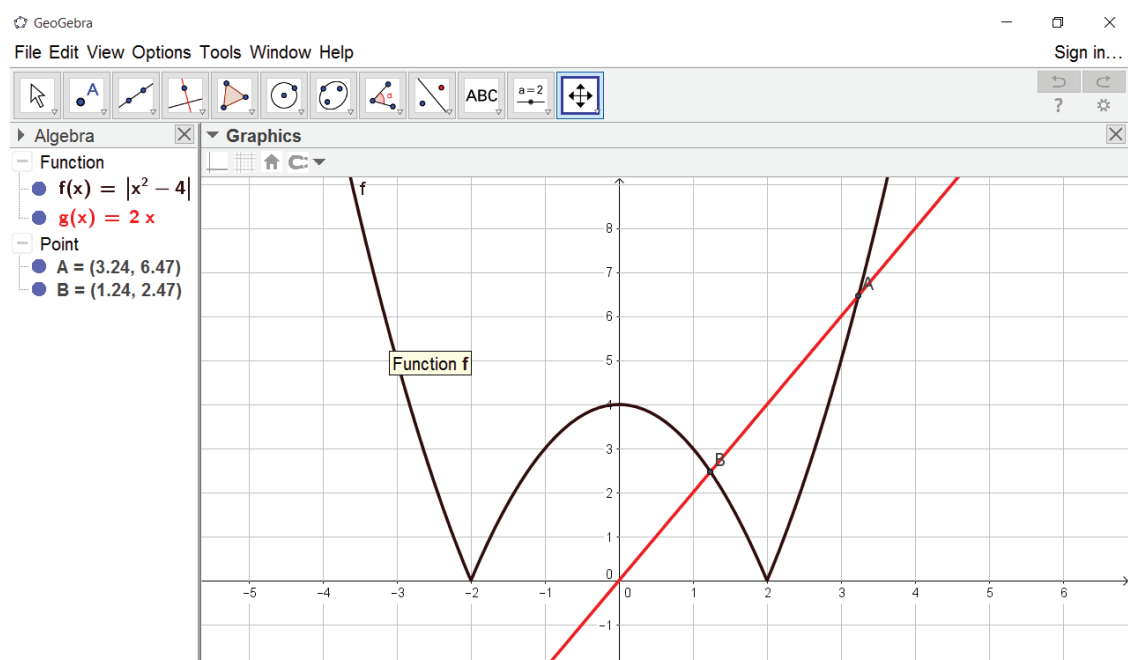
and therefore there was no solution. The illustration in GeoGebra, brought out a learning point, not only to Peter but to the rest of the participants. The fact that a modulus function cannot be equal to a negative constant is brought to fore. From the illustration, Peter concluded that there was no solution to that equation.

Fig 4.37: The graph of $|x - 2| = -3$.



Solving the equation $|x^2 - 4| = 2x$

Fig 4.38: Using GeoGebra to solving $|x^2 - 4| = 2x$.



Sophie started her solution by illustrating the two functions $f(x) = |x^2 - 4|$ and $y = 2x$ in GeoGebra (see Fig 4.38). She then used the algebraic method to solve the equation. She restricted her calculation to only two cases. Insert 4.6 shows Sophie's answer.

Insert 4.6

Handwritten algebraic solution on grid paper:

$$|x^2 - 4| = 2x$$

$$x^2 - 4 = 2x$$

$$x^2 - 2x - 4 = 0$$

$$x = 3.24 \text{ or } x = -1.24$$

$$x = 3.24 \text{ at Point A}$$

OR ~~$x^2 - 4 = -2x$~~

$$x^2 - 4 = -2x$$

$$x^2 + 2x - 4 = 0$$

$$x = 1.24 \text{ or } x = -3.24$$

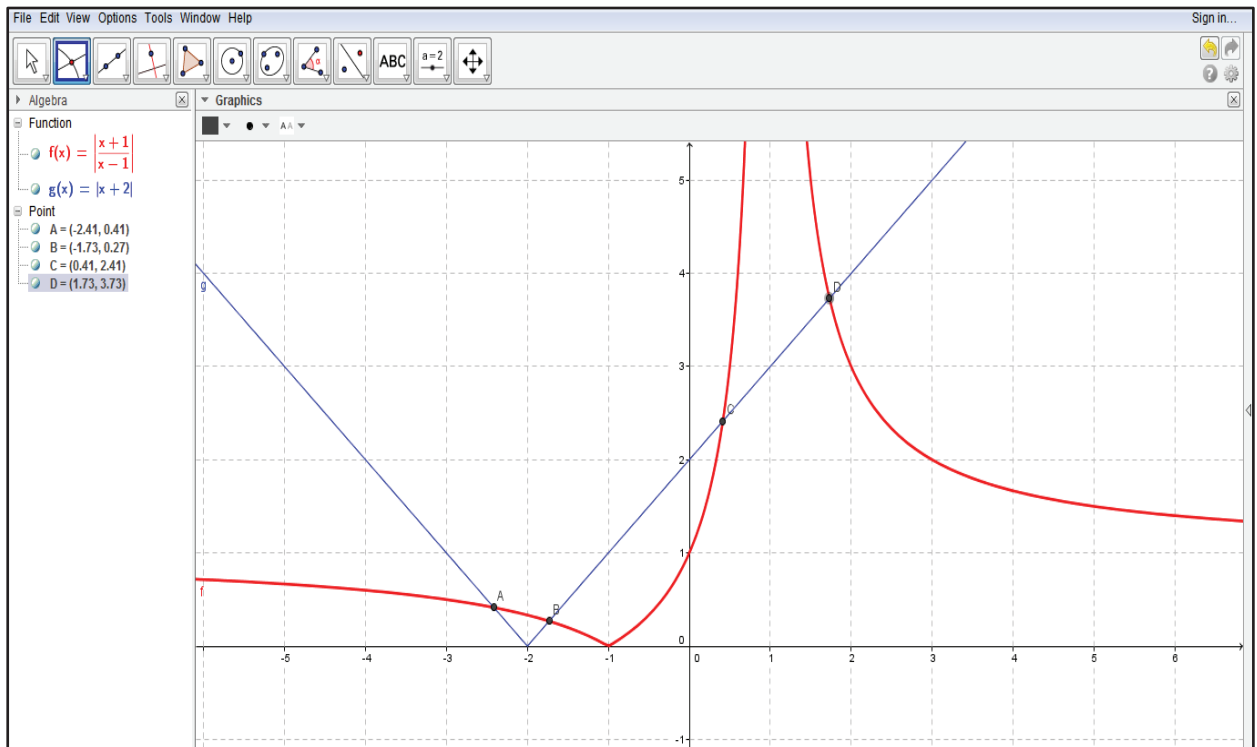
$$\therefore x = 1.24 \text{ at Point B}$$

GeoGebra provided Sophie with a clear view of the points of intersection and limited her algebraic calculations to focus on the parts of the graphs that were intersecting.

Extending work to included algebraic fractions $\left| \frac{x+1}{x-1} \right| = |x + 2|$

Smart (1995) cited in Selinger and Pratt (1997) found that while using graphic calculators, students unusually displayed high levels of visual strategies to solve algebraic problems. None of the six participants had done any Further Pure Mathematics and they had not encountered graphs of rational functions. However, they used GeoGebra to identify the solutions. Participants sketched the graphs of $f(x) = \left| \frac{x+1}{x-1} \right|$ and $g(x) = |x + 2|$ on the same diagram. Below is an extract of a diagram captured from Susan and Peter's screencast clip.

Fig 4.39: Solving $\left| \frac{x+1}{x-1} \right| = |x+2|$ graphically.



The multiple representations identified points A, B, C and D in the graphics section and corresponding coordinates for each point were displayed in the algebraic section of the screen. Solutions listed in the algebra register are $x = -2.41$; $x = -1.73$; $x = 0.41$ and $x = 1.73$. The visual representations in GeoGebra guided participants in solving this equation. No one in the group attempted to solve this equation algebraically. Susan and Peter passively accepted answers as given on the screen and ended the investigation. No curiosity was raised and no discussion was generated from these displays.

Illustrations in GeoGebra (Fig 4.38 and Fig 4.39) clearly showed how many solutions there are, before the start of the algebraic calculations. This restricted participants to work on parts of the graphs within the correct domains. GeoGebra afforded participants an opportunity to visualise the notions of different types of equations of modulus functions and helped them redefine the domain and range of the functions.

The visual illustration established further properties of the domain and range of the graphs of modulus functions. In cases where the range of the function was not first

established, as visualised on the graphs in Fig 4.35, there are clearly four points of intersection, two of which lie outside the range. Points C and D lie outside the domain of the defined modulus function, hence the solutions at these points should be discarded.

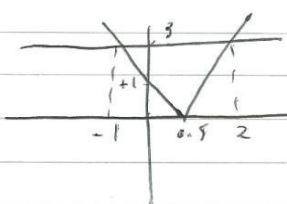
Solving equations involving functions graphically was mostly a method of last resort, for the pairs of Susan/Peter and John/Emma, when the algebraic method became complicated or had failed. It was not a method of first choice. In conclusion, to all the activities in Session 4, I recommended to participants to sketch the graphs first, to see the number of solutions and identify where they are located.

4.5 SESSION 5: Solving inequalities involving modulus functions.

By the time we started working on inequalities involving modulus functions, the use of GeoGebra had become less and less. Research on dynamic mathematics software, (Guin & Trouche, 1999; Hoyles, Noss & Kent, 2004; Mariotti, 2002; Trouche, 2005), notes that the use of multiple representations and accurate diagrams drawn using dynamic mathematics software shapes learner's mental conception of function graphs. Using multiple representations in GeoGebra was presumed to have helped participants to build mental pictures of what linear and quadratic graphs of $|f(x)|$ look like (see pencil and paper sketches).

Insert 4.7: Solving the inequality $|2x - 1| \leq 3$

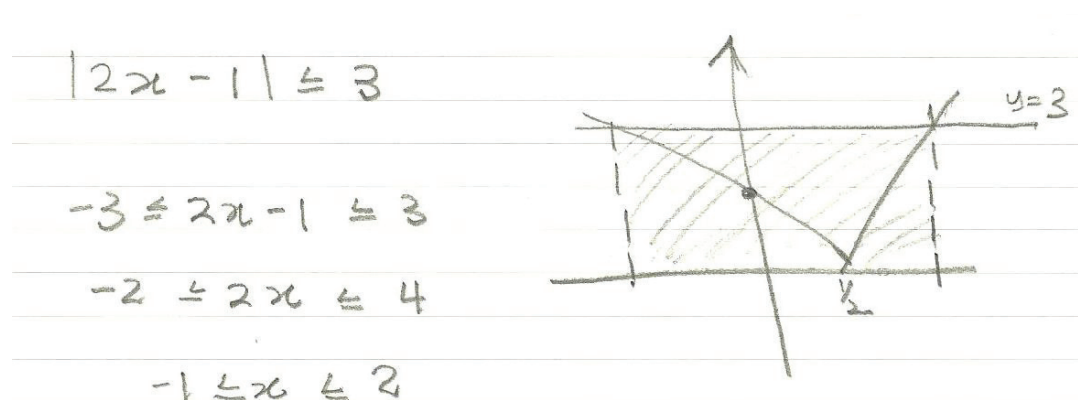
Below is an extract from Peter's pencil and paper work.

<p>1) $2x - 1 \leq 3$.</p> <p>$2x - 1 = 3$</p> <p>$2x = 4$</p> <p>$x = 2$</p> <p>$-(2x - 1) = 3$</p> <p>$2x - 1 = -3$</p> <p>$2x = -2$</p> <p>$x = -1$</p>	 <p style="text-align: center;">$-1 \leq x \leq 2$</p>
--	---

Peter sketched the inequalities on a diagram and then proceeded to use the algebraic method to get the solution. Peter used equality signs throughout but recognised in the final solution that the graph of $y = |2x - 1|$ is below the graph of $y = 3$ between -1 and 2 and gave the solution in the correct format. The pencil and paper sketch is not as accurate as those produced by GeoGebra, but it however gave him guidelines on how to solve the inequality $|2x - 1| \leq 3$, by identifying the range of values for which the graph of $y = |2x - 1|$ was below that of $y = 3$.

Insert 4.8: Solving the inequality $|2x - 1| \leq 3$ from Susan's pencil and paper work

Susan also started the solution by using a sketch.



It was interesting to note that Susan and Peter began their solution process with a visual justification using pencil and paper sketches and then went on to use the algebraic methods to solve inequalities. Following recommendations from Session 4, Susan and Peter's solutions were a combination of the visual aspect on the pencil and paper sketch and an algebraic method based on the graph of the modulus function. Even though they used visual images to solve the inequalities, their dependence of GeoGebra had become less. They sketched the graphs involving linear functions easily without the aid of the software.

The algebraic and graphical representations observed throughout the activities are two very different symbol systems that articulate and define the mathematical concept of modulus functions. Neither functions nor graphs can be treated as isolated concepts. These are two symbolic systems used to illuminate each other. They both contribute to and confound the development and understanding of modulus functions.

4.6 SESSION 6: Participant responses at the end of the session.

The last session checked on the impact of GeoGebra, and how participants used knowledge acquired through using the software, to tackle examination type questions, sketch graphs, solve linear equations and inequalities involving modulus. The discussions were guided using an interview protocol (see chapter 3).

Interview data revealed that participants had a positive impression of using GeoGebra. Participants' interest was further demonstrated in the way they were deeply engrossed in trying different tools, some of which were not included on the initial list discussed in Table 4.1.

James and John said it was easy to navigate around GeoGebra than on their graphics calculators in that they could move from one function to another without having to turn over the screen. This observation resonates with the rest of the participants and also concur with Dikovic's (2009) that GeoGebra is more user friendly compared to graphics calculator, and that it offers an easy-to-use interface, multilingual menus, commands and help lines.

4.7 Summary

The chapter highlighted the participants' progression in using GeoGebra and learning the concept of modulus functions. It gave an account of activities carried out during the investigations. The account is drawn from lesson observations and screencast video recordings of activities from across the six sessions. The activities were made up of sequentially planned lessons conducted with a group of six participants. For each of the sessions, raw data collected from video recordings was collated, coded and analysed. Data was presented in Figs, Vignettes, Tables and Appendices.

The first section of this chapter gave an account of participants' instrumental genesis process as they mastered the use of GeoGebra to explore the concept of modulus functions. It highlights participants' activities while investigating graphs and also solving equations and inequalities involving modulus functions.

In some sections, the account highlights the use of multiple representations to link algebraic functions and their graphic representations. Though most activities looked at

finding and describing the graphical representations of modulus functions, there are activities when participants were required to convert given graphical representations to algebraic expressions. Analysis of participants' work, shows evidence of understanding of modulus functions even in cases when they had stopped using GeoGebra.

The highlights of the chapter include the discussion the distinction between turning and stationary points; the difference in the graphs of $y = |f(x)|$ and $y = f(|x|)$; the graphs of $y = \ln(x)$ and $y = \ln|x|$ which generated a lot of discussion on why the integral of $1/x$ is $\ln|x|$ instead of $\ln(x)$. We also discussed the issues of domain and range of function to avoid getting spurious answers when solving equations and inequalities involving modulus functions.

CHAPTER 5

5.1 Conclusions and Discussions

In this chapter I used information from Chapter 4 to answer the research questions. Each question is addressed based on the data available. Analysis of data in Chapter 4 highlighted some aspects that I had not anticipated and it helped me to ask some probing questions throughout the sessions. I used Goos et al's (2004) four metaphors to analyse how participants used GeoGebra and at what stage in the process of solving problems did they use it. I checked for fluency in using the software and students' comments about the software. These observations, together with data collected from other sources such as pupils recorded discussions, screencast video activities and pencil and paper work, established whether students used GeoGebra to investigate the modulus function concept or to verify answers or as an extension to investigate more advanced aspects of modulus functions and their graphs.

Initial observations show that the use of GeoGebra was very frequent in the first four of the six sessions. The high frequency of use of GeoGebra within the first two sessions can be attributed to the instrumental genesis process, with work mostly focused on GeoGebra than on modulus functions. Some time was spent while participants tried to identify the correct syntax.

In later sessions, the use of GeoGebra depended on the need to investigate different aspects of modulus functions. There was reduced frequency of use of GeoGebra in Session 4 and 5 when participants were solving equations and inequalities involving modulus functions. This could have been influenced by the fact that: i) participants had internalised the graphical representations of modulus functions and could now sketch some of them without the aid of GeoGebra, (ii) the structure of some examination questions did not provide for the use of GeoGebra or graphics calculators (see Appendix D).

Participants used GeoGebra to investigate functions that would have been difficult when using pencil and paper (polynomials, algebraic rational fractions and sum of modulus functions). This was evident when participants used the calculus to investigate modulus

functions. Calculating derivatives of modulus functions is beyond the scope of the Core 3 curriculum specifications in England and Wales (see Appendix H). This was an extension to their understanding of modulus functions.

5.2 Findings:

5.2.1 Primary Research Question 1: *In what ways did participants employ GeoGebra to support their understanding of the concept of modulus functions?*

Within an instrumental genesis approach, I used Goos' et al. (2004) four metaphors to answer the first research question. The instrumental approach focuses on the way individual learners use tools. My research looked at group interactions and whole class discussions while participants used technology to share and test their mathematical understanding, therefore Goos et al.'s metaphors were appropriate to answer this research question.

5.2.1.1 Research Question 1: Metaphor 1:

According to Goos et al (2004), at the first level of use, students view technology as a master. Their knowledge and usage of the technology is limited to a narrow range of operations.

An analysis of data from Activity 1 of Session 1 concurs with the first metaphor. Participants carefully used basic GeoGebra tools introduced at the start of session 1 (See Table 4.1). The use of GeoGebra was restricted to simple tasks. Participants appeared to be satisfied with inputting as many functions as they wanted, but sticking to the basic tools. There was not much debate, throughout the sessions, about whether graphical presentations shown on the screens were correct or wrong except in Session 3, Activity 2, when there was a discussion about the correctness of the gradient function graph for $\ln(x)$ (see Vignette 14). This was highlighted again when participants were introduced to sliders. They used the default setting on the screen without adjusting, for example, intervals to give natural number values for parameters a , b and c .

5.2.1.2 Research Question 1: Metaphor 2:

At the second level, students use technology as a supplementary tool but not in creative ways to change the nature of activities (Goos et al 2004).

The nature of the exercise in Appendix E allowed participants to use GeoGebra, only to check their answers. The exercise consisted of a set of graphs and algebraic functions. Participants were matching the graphs and the corresponding functions. After matching the graphs and their functions, participants entered the functions in GeoGebra to check if their match was correct. There was not enough evidence from the data to conclude that participants used GeoGebra as a supplementary tool after the matching, or that they first entered the functions in GeoGebra and then selected the corresponding image.

5.2.1.3 Research Question 1: Metaphor 3:

At the third, level students use technology creatively to provide new ways of approaching existing tasks (Goos et al 2004, Templer, Klung and & Gould 1998).

Characteristics of this level of usage were highlighted throughout each session. Starting from Activity 2 of Session 1, participants began to explore more features of modulus functions. They used sliders and animations to investigate features of modulus functions. In Activity 2, participants discovered new ways of creating sliders without guidance from me. In Fig 4.11 participants introduced a straight line $y = 4$ to check the effects of changing values of a , b and c in the function $y = |ax + b| + c$. In Fig 4.26 participants investigated the gradient for the derivative of $\ln(|x|)$. All this extended beyond the basic investigation of graphs of modulus functions.

In Session 2, Activity 1 (see Fig 4.13) there was a slight diversion from the planned tasks, when participants used calculus to investigate turning points. In Activity 3 of Session 3 participants ventured to investigate graphs of the sum of modulus functions. The discussions that followed did not indicate that participants fully understood the graphs that were displayed. However, that did not scare them from attempting even more complicated sums of modulus functions as in Fig 4.32 and Fig 4.33.

5.2.1.4 Primary Research Question 1: Metaphor 4:

In the fourth and the last level, students use technology as extension of self. At this level, students use technology to extend their existing competencies related to mathematics and the technology (Goos et al 2004).

Evidence from the screencast video recordings highlighted incidences where participants used GeoGebra to extend their understanding of modulus functions. Some of which are:

1. Distinguishing the difference between $y = |f(x)|$ and $y = f(|x|)$ and being able to sketch the two graphs without the aid of GeoGebra (Session 3 Activity 3 Questions 1 – 5);
2. Raising the question why the integral of $1/x$ is $y = \ln |x| + c$ instead of $y = \ln(x) + c$ since the derivative of $f(x) = \ln(x)$ is $1/x$ (Vignette 14).
3. Using GeoGebra to eliminate spurious answers derived from algebraic calculations while solving equations (Session 4, Activity 1: Fig 4.34 – Fig 4.39) and inequalities that involve modulus functions.
4. The concept of continuous functions and derivatives was highlighted when participants tried to construct tangents at the point where the graph of the modulus function was reflected in the x-axis (Session 2, Activity 3).

5.2.1.5 Primary Research Question 1: Mediating tool:

Outside Goos et al's metaphors, GeoGebra was also used to mediate in the learning of modulus functions. Central to understanding how tools mediate learning is the need to understand how students' mathematical conceptions were shaped through the use of GeoGebra. GeoGebra acted in a mediating role by providing a medium for visualisation that linked the abstract concept of modulus functions with graphical illustrations. It afforded an environment in which quick and precise diagrams were made. This is evident in participants' work throughout the sessions when they used GeoGebra to generate graphs of modulus functions involving simple linear functions, quadratic functions, trigonometric functions, and exponential and logarithm functions.

There was a need to analyse how mathematical learning evolved in tandem with the use of GeoGebra. Brousseau (1997) and Pea (1997) argue that a mere action of pointing, clicking and dragging the slider buttons allows mediation of learning as learners try to make sense of changing representations in the different multiple representation formats.

There were noted instances where pupils used GeoGebra as a mediating tool in their learning. One such instance was when participants struggled to use the correct syntax to input functions in GeoGebra. There was dialogue as participants developed new phrases to describe the shape of the graph of the modulus function. The use of descriptors such as, 'V shaped' to refer to the shape of modulus function graphs and emergence of terms like corners, reflection, translation, domain and range were all part of an endeavour to come up with a formal description of graphs of modulus functions (see Table 4.2).

Another instance of tool mediation was when participants appeared to understand a particular aspect of GeoGebra but in fact had entirely their own perspective. This was manifest in the misconception that all turning points on the graph of a function were stationary points. In Session 2, Activity 1, Sophie and James had the misconception that the graph of $y = |x^2 - 1|$ had three stationary points at $x = -1$, $x = 0$ and $x = 1$. The use of an animated diagram using the tangent tool in GeoGebra (Session 2, Activity 3) and calculus (Vignette 10) cleared this misconception.

In Fig 4.18, the derivative of $f(x) = |x^3 - 5x^2 + 2x + 8|$ and the sum of two modulus function graphs in Fig 4.32 and Fig 4.33, are displayed graphically. Participants failed to relate the graphical representations for these functions to the graphs of modulus functions they had previously encountered. More detailed critical analysis of the conversations revealed that participants failed to grasp the connections between the algebraic and graphical representations as displayed on the screens. They just accepted each given representation as correct. Similarly, Yerushalmy (1991) found that even with extensive experience with multiple representational learning experiences designed to understand functions, students still experienced problems in connecting between graphical and algebraic representations. An investigation by Ainsworth, Wood

and Bibby (1996) demonstrated how translation between graphical and algebraic representations varied, depending upon the nature of the function and the type of representation selected.

Another illustration of mediation of learning is how earlier experiences of successfully constructing graphs of modulus functions tended to structure later constructions when participants used pencil and paper without the aid of GeoGebra. There was evidence, found at the end of Session 1 activity 3, on how earlier experiences with GeoGebra structured participants' later construction of graphs of modulus functions. A task of simple recognition, without involving construction of modulus function graphs was given. Participants were able to recognise correctly the modulus functions that represented each of the given graphs (see Appendix E). In another instance in Activity 3 of Session 3, most of the participants managed to answer questions selected from past examination papers correctly without the aid of GeoGebra (see Appendix D). An analysis of participants' conception of modulus function graphs at the beginning of Session 1 and participants' written answers from Appendix B and C showed a progression in participants' understanding of modulus function graphs.

5.2.1.6. Secondary Research Question 1:

Does the use of GeoGebra increase speed and efficiency of the process of sketching modulus function graphs? Does it improve the accuracy and presentation of results?

Analysis of the data collected indicated that within the first session of this investigation, participants managed to acquire skills of using basic tools in GeoGebra while at the same time investigating graphs of modulus functions. GeoGebra made it easier to produce modulus function graphs accurately and rapidly, therefore increasing the efficiency and speed at which the work was covered throughout the sessions. The graphical screen of GeoGebra displayed graphs, which enabled participants to focus on deducing properties of the graph of $y = |f(x)|$ instead of worrying about manually sketching the graphs.

Work covered in Session 1, Activity 1 indicated that participants made a solid start to the development of a mental picture of concepts related to modulus functions and their

graphs, within a short space of time as evidenced by the work displayed in Figs 4.1 – 4.7. After the first session, all participants were able to explain correctly how to sketch the graph of $y = |f(x)|$ from the graph of $y = f(x)$.

Participants showed a more immediate and practical grasp of modulus functions and the software (GeoGebra) as they progressed through the work. This was reflected in participants' answers to past examination questions presented at the end of Session 3 Activity 2, where participants managed to sketch graphs of the type $y = |f(x)|$ and $y = f(|x|)$ from a given graph of $y = f(x)$, without the aid of GeoGebra.

Speed and efficiency of the software was manifest in the variety and number of functions investigated within a short space of time. By the end of the first session, participants had managed to extend their work beyond linear functions, to include trigonometry functions and quadratic functions involving modulus functions. In session 2, participants looked at polynomials of a degree higher than 2, exponentials and logarithms, and still had time to discuss differentiation of modulus functions. Within a traditional approach more time is spent on investigating graphs of linear functions involving modulus functions with the hope that the properties will be cascaded to graphs of non-linear functions.

Participants agreed (see Table 4.2) that the easy input and instant displays generated by GeoGebra gave them access to understanding modulus functions and their properties quickly. Participants also observed that fast and accurate constructions in GeoGebra saved them time, which was then devoted to carrying out further investigations, e.g. attempting more advanced functions like the sum of two modulus functions, $y = |ax + b| + |cx + d|$ (see Fig 4.33).

Efficiency was manifest in the number of times participants diverted from the main investigation to follow issues that emerged from the multiple representations. Vignette 10 highlighted instances when participants investigated the derivative of modulus functions and in Vignette 14 participants compare the graphs of $y = \ln(x)$ and $y = \ln(|x|)$ and even question the validity of the integration of the function $1/x$, otherwise taken for granted.

5.2.2 Research Question 2: How does GeoGebra relate to and contribute to the whole learning experience?

Question 2 was divided into three sub-research questions and each of these is discussed separately based on the data collected through screencast video clips and field notes.

5.2.2.1 Is there any evidence of participants using GeoGebra to support processes of checking, refining, testing and improving strategies and solutions?

GeoGebra made work on modulus functions more accessible to participants. It supported the processes of checking, trailing and refinement through the use of multiple examples. Instances of GeoGebra supporting the process of checking, refining, testing and improving strategies were evident in participants' work throughout the study. From the first session participants used as many sketches as they could, to establish common characteristics of the graphs of modulus functions and this is highlighted in Table 4.2. Sliders enabled participants to dynamically manipulate representations of modulus functions and observe how the properties of the functions changed, leading to formulation of formal descriptions in Vignettes 7 - 9. In one of the exercises (see Appendix B, Questions 1 - 9), participants attempted questions on the grid provided, then used GeoGebra to mark their own work.

Different approaches to solving problems were observed in Session 2 when participants used GeoGebra to check the derivative of modulus functions. When participants were confronted with unfamiliar graphs during the investigation of derivatives of modulus functions, they used GeoGebra to search for a clear understanding of what was taking place. In solving equations involving modulus functions, participants realised through the use of GeoGebra, the need to define domains of functions to avoid getting spurious answers.

5.2.2.2 Does the use of GeoGebra overcome participants' difficulties and enhance their sense of capability and confidence to tackle difficult tasks?

Participants' autonomy seemed to increase as they became more familiar with GeoGebra and they developed the confidence to try things out in an experimental manner. They were motivated to seek justifications for their descriptions of graphs of modulus functions. Smart (1995) cited in Selinger and Pratt (1997) in a research that focused particularly on girls' mathematical development, observed that visual skills increased significantly alongside development of confidence in mathematics. Data from Session 2 and Session 3 showed that participants gained confidence, to attempt more advanced functions, as they became familiar with GeoGebra. They investigated functions that would normally be ignored when using pencil and paper. These included algebraic fractions involving modulus functions. It was not easy at times to get the correct syntax, but from the perseverance shown in the video clips, it showed participants' sense of confidence in using GeoGebra.

Reduction of laborious written work enabled participants to work through a lot of examples and allowed them to tackle issues beyond the tasks of sketching and solving problems involving modulus functions. GeoGebra widened the range of possible activities and provided deeper reflection, exploration and problem solving than in a pencil and paper environment. The dynamic connection between multiple representations of mathematical objects seemed to open up a range of capabilities of GeoGebra software, which can be used for teaching and learning mathematics while fostering student understanding (Preiner, 2008). GeoGebra provided multiple representations which have been strongly connected with the complex process of learning in mathematics, and more particularly, with the seeking of the students' better understanding of important mathematical concepts such as graphical representation of modulus functions.

Evidence of capabilities to tackle more mathematics concepts was manifest in some of the issues that emerged during the investigations: (a) differentiability versus continuity of functions, when participants were investigating turning points of graphs of modulus functions (see Vignette 10), (b) Turning points and stationary points; (c) the nature of

graphs of gradient functions (See Figs 4.14 – 4.18), (d) importance of setting the domain and range of functions, which led to the need for knowledge of a more formal definition of a function, and (d) differentiation of $\ln(x)$ and the reverse process of integrating the reciprocal function $y = 1/x$.

5.2.2.3. *Does the use of GeoGebra foster participant independence and peer exchange, notably by providing opportunities for pupils to share knowledge and provide mutual support to peers?*

A few approaches to learning were observed as participants worked with GeoGebra. Firstly, there was an ‘independent approach to learning’. Participants managed to explore GeoGebra (the software) without much assistance from me. Participants independently explored the use of GeoGebra and developed enough proficiency to use the software to investigate graphs of modulus functions. They explored the tools menu, they investigated how to input information and they made remarks about the multiple representations on a GeoGebra screen. Participants did not view me as the source of information. When they struggled to input commands in GeoGebra, they consulted their peers or searched for any leads from “Input help” facility located at the end of the Input window.

GeoGebra provided participants with immediate feedback and fostered peer discussions focused on multiple representations on the screens. Immediate feedback allowed participants to move on with their work uninterrupted, hence fostering independency. Research literature (Clements, 2000), shows that dynamic mathematics software provides fast and reliable feedback, which is non-judgemental and impartial. Clements (2000) argues that working in real-time encourages students to make their own conjectures, test them out and modify them.

Evaluation of participants’ work from screencast video clips showed that they carried out investigations using a trial and error approach in most instances. Participants used many entries until they observed similar characteristics. They synthesised all graphical representations of modulus functions and noted the difference between the graphs of $y = |f(x)|$ and $y = f(|x|)$. Descriptions such as reflection of the negative part of the graph of y

= $f(x)$ in the x-axis, to sketch the graph of $y = |f(x)|$ and reflection of the graph of $y = f(x)$ for $x \geq 0$ in the y-axis, to sketch the graph of $y=f(|x|)$ are noted (see Fig 4.20).

5.3 Discussions

This research study showed that multiple representations based instruction to learn about modulus functions, made a significant influence on participants' understanding of this concept. In the initial stages of the investigations, learning of modulus functions using GeoGebra was fraught with difficulties, requiring participants to relate graphs to corresponding algebraic representations. As work progressed, participants developed conceptual understanding between some of the graphical representations offered by GeoGebra and their algebraic alternatives.

In these investigative activities, participants tested and refined some of the conjectures that they had made during the first session of the study. Based on an analysis of the responses to past examination questions, most of the participants seemed to have made a good start to the development of conceptual schema of concepts related to modulus functions. In question 1 (see Session 3, Activity 3) all participants managed to correctly sketch the graphs of $y = |f(x)|$ and $y = f(|x|)$.

Data from lesson activities revealed that participants worked faster in the last lesson as they got more familiar with GeoGebra. Participants gathered a lot of sketches that characterised the graphs of modulus functions and managed to synthesise the gathered diagrams and made connections between (i) the graphical representations of $y = |f(x)|$ and $y=f(|x|)$, (ii) the relationship between functions and their derivatives and (iii) the link between the graphs of functions and their derivatives.

Participants' understanding of graphs of modulus functions was improved by working with GeoGebra. After a slow start due to coding problems, participants generated many graphs of modulus functions. The use of different colours for different functions made it easy and clear to display many graphs on the same screen and observe common characteristics. GeoGebra enabled participants to link the algebraic concept of functions and their graphical representations.

My own observations based on the video recordings and classroom observations reveal that participants became motivated while using GeoGebra and they remained within the context of investigating graphs of modulus function. Focus on activities was evident from the way participants persevered when faced with difficulties of finding the correct command to input $y = |f(x)|$ in GeoGebra and the number of sketches generated to compare the relationship between the graphs. When error messages were displayed on the screen, participants tried other means to find the correct notation.

Participants' conversations were more focused on finding general descriptions about the behaviour of the graphs of $y = |f(x)|$ and $y = f(|x|)$ (see Vignette 12 – 14). GeoGebra afforded them the opportunity to use as many graphs as they wanted without any challenges on how to sketch them. This enabled participants to try a variety of options and compare results, make generalisations and move towards a more formal description of how to sketch the graphs of $y = |f(x)|$ and $y = f(|x|)$ from the graph of $y = f(x)$.

CHAPTER 6:

6.0 Contributions and implications of the study:

In this research, I used two theoretical frameworks (Instrumental genesis and semiotic mediation) to investigate how GeoGebra was used and how it was transformed by participants during activities while learning the concept of modulus functions. For this I focused on Goos et al's (2004) metaphors to elaborate instrumental genesis within a group context and multiple representations within a semiotic mediation framework.

One novel contribution of this research is its elaboration on Goos et al's (2004) framework within the context of learning about modulus functions using dynamic mathematics software (GeoGebra). It draws data from students' experiences in a real classroom setting, where they engaged technology to mediate in the process of understanding modulus functions.

The results of this study showed that GeoGebra-assisted instruction, as a supplement to traditional classroom instruction, allowed participants to explore more aspects of modulus functions than the confines of the Core 3 module specifications (see Appendix H). My findings are consistent with studies by Mainali & Key (2012), Lowrie (2001), Ruthven et al (2008), Ainsworth et al (2002), Brenner et al (1995), Ozgun-Koca (2001) and Panasuk (2010), who found positive impact of utilising mathematical software to enhance students' learning and understanding.

A study conducted by Guzen and Kosa (2008) among mathematics teachers in Turkish schools, also showed that computer supported activities contributed to the learning and understanding of mathematical concepts. My study demonstrated the instructional effectiveness of GeoGebra to support the learning of modulus functions. It challenges some advocates of a constructivist approach to education who argue that dynamically linking representations through the use of technology leaves learners too passive in the process (Ainsworth 1999).

This study also contributes to existing research literature (Drijvers, Kieran & Mariotti, 2010; Arzarello, 2006; Falcade et al., 2007; Bartolini Bussi & Mariotti 2008; Duval, 2006)

on instrument-artefact as a semiotic system in the wider sense of the term. It focused on how dynamic mathematics software (GeoGebra) was transformed into an instrument by participants and how it assisted their learning within the novel context of learning modulus functions.

GeoGebra was used as a semiotic mediation tool in the learning process to provide visual representations that enhanced understanding. It complemented activities during regular classroom settings where students were able to visualise the same concept in multiple representational formats. The multiple screen representations in GeoGebra brought together three aspects of modulus functions (the function, the graphical representation and the equation) which are normally taught separately in the school curriculum in England. It enabled participants to visualise all three aspects at the same time, hence promoting conceptual understanding of the concept of modulus functions, a finding similar to Brenner et al (2005). In solving equations and inequalities involving modulus functions, in session 5 and 6 participants began some of their solutions with a visual justification using pencil and paper sketches and then went on to use the algebraic method to solve the equation or inequality.

My research conclusions are consistent with the findings of previous studies (Ozgun-Koca, 2001; Pitts 2003) that provide evidence to the effectiveness of multiple representations based instruction in engaging students in meaningful algebra learning. After multiple representation based instruction in a college algebra course, Herman (2002) cited in Akkus et al (2009) found that students were better able to establish connections between varieties of representational modes, which led to better understanding of algebra. Herman's (2002) research findings challenge some advocates of a constructivist approach to education who argue that dynamically linking representations through use of technology leaves a learner too passive in the process (Ainsworth 1999).

Dynamic links provided by GeoGebra encouraged active participation and reflection on the nature of modulus function, leading participants to construct the required understanding of the concept. Participants saw things dynamically changing and understand how they were.

6.1 Limitations of the study:

This study adds to literature on the use of new technologies by investigating students' experiences of using GeoGebra to learn modulus functions. The basic limitation of this research was that it investigated a single instrument, GeoGebra, with a small group of participants, over a very short period. The study occurred in a particular time and was restricted to particular curriculum requirements. The results could have been more reliable if the study had been conducted in more places, with more than one researcher and more participants. Although the study cannot be generalised to an overall population because of the small number of research participants, the observations concur with previous studies on other dynamic mathematic software like Dynamic Geometry Software, Cabri and Derive.

At the start of this research, my knowledge of GeoGebra was limited to a few sessions on personal use and a few instances where I used animated GeoGebra applets to demonstrate transformations of the graph of $y = f(x)$ to $y = f(x \pm a)$; $y = f(ax)$; $y = f(x) + a$ and $y = af(x)$. Although I managed to assist participants through the investigations, it would have been more helpful in my planning, if I had taken more time to use the software before the start of the study. I discovered some of the functions of the software along with the participants.

Throughout my thesis, I strove to assert great caution while conducting an interpretive phenomenological analysis of data and not to focus only on positive aspects of the software. During transcription and analysis of data I tried to maintain trustworthy and credibility by presenting accounts of participants' experiences as viewed from the clips. I presented transcribed data to participants to check if the transcripts were a true reflection of what had transpired during the learning activities.

6.2 Recommendations for further research:

The results of this study showed that there is great potential in using GeoGebra to teach 'A' level mathematics in England and Wales, but further research is necessary to extend the investigations to more topics across the mathematics curriculum. Research on teachers' professional development in order to use GeoGebra effectively in the classroom is needed. It is necessary to continue addressing issues such as: the general

impact of GeoGebra on mathematics learning, the variety of approaches to GeoGebra and the use of GeoGebra to develop an understanding of mathematical ideas. Finally, a study on the effects of the use of GeoGebra in students' mathematical attainment and achievement needs to be carried out over a longer period than that of this study.

REFERENCES:

- Ainsworth, S. (1999). The functions of multiple representations. *Computers and Educations*
Accessed online: Url: <http://www.psychology.nottingham.ac.uk/staff/sea/functions.pdf>.
Date: 12/07/2013.
- Akkus, O. and Cakiroglu, E. (2009). The effects of multiple representations-based instruction on Seventh Grade students' algebra performance. In *Proceedings of CERME 6, January 28th – February 1st 2009*. Lyon France [Accessed on line October 2014. Url: www.inrp.fr/editions/cerme6.
- Attorps, I., Bjork, K. & Radic, M. (2011). The use of mathematics software in university mathematics teaching. *Paper presented at CERME7, The Seventh Congress of the European Society for Research in Mathematics Education*, 9th- 13th February 2011. Rzeszow: Poland.
- Artigue, M. (1998). Teacher training as a key issue for the integration of computer technologies. In Tinsley, D. & Johnson, D. (eds.) Vol. 119, *Proceedings of the IFIP TC3/WG3.1 Working Conference on Secondary School Mathematics in the World of Communication Technology: Learning, Teaching, and the Curriculum: Information and Communications Technologies in School Mathematics*. London: Chapman and Hall.
- Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, 7(3), 245-274.
- Atkinson, J. M., & Heritage, J. (1984). *Structures of social action: Studies in conversation analysis*. Cambridge: Cambridge University Press.
- Arzarello, F.; Olivero, F.; Paola, D., & Robutti, O. (2002). A Cognitive Analysis of Dragging Practices in Cabri Environments. *ZDM*, 34(3), 66-72.
- Arzarello, F. & Edwards, L. (2005). Gesture and the construction of mathematical meaning. In H. L. Chick & J. L. Vincent (Eds.). *Proceedings of the 29th Conference of IGPME*, 1, 123-154. Melbourne: PME.
- Arzarello, F. (2006). Semiosis as a multimodal process, *Relime Vol Especial*, 267-299.
- Bartolini Bussi, M. G. Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: artefacts and signs after a Vygotskian perspective. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of International Research in Mathematics Education (2nd ed., pp. 746–805)*. Mahwah: Lawrence Erlbaum.

- Bartolini Bussi, M.G.; Mariotti M.A.; & Ferri, F. (2005). Semiotic Mediation in the Primary School: Durer's Glass. In H. Hoffmann, J. Lenhard & F. Seeger, *Activity and Sign – Grounding Mathematics Education* (Festschrift for Michael Otte), 77-90 New York: Springer
- Bartolini Bussi, M. (2001). The geometry of drawing instruments: arguments for a didactical use of real and virtual copies, *Cubo Matematica Educacional*, 3(2), 27-54.
- Bayazit, I. and Aksoy, Y. (2007). *Connecting Representations and Mathematical Ideas with the use of GeoGebra: The Case of Functions and Equations*. Ecriyes University: Turkey.
- Becta (2004). *A review of the research literature on barriers to the uptake of ICT by teachers*. Coventry: BECTa
- Becta (2003). *What the research says about using ICT in Maths*. UK: BECTa ICT Research.
- Becta (2007). *Harnessing Technology Review 2007: Progress and impact of technology in education*. Coventry: BECTa
- Bell, A. W.; Swan, M.; Onslow, B.; Pratt, K.; Purdy, D. (1985). *Diagnostic teaching: Teaching for long term learning*, Report of ESRC Project HR 8491 / 1, Nottingham, UK: University of Nottingham, Shell Centre for Mathematical Education.
- Bell, A. (1993). Some experiments in diagnostic teaching, *Educational Studies in Mathematics*, 24, 115-137.
- Bell, J. (2006). *Doing your Research Project: A guide for first-time researchers in education, health and social science* (4th Edition). Open University Press. Maidenhead: England
- Bereiter, C. and Scardamalia, M. (2005). *Technology and literacies: From print literacy to dialogic literacy*. Toronto: Ontario Institute for Studies in Education. [Online line. <http://ikit.org/fulltext/TechandLit.htm>. Accessed on 11/06/2012]
- Bismarck, S. (2009). *Mathematics teacher roles when using technology: Understanding the roles of facilitator and mediator*. A Dissertation submitted to the Graduate Faculty of the University of Georgia in Partial Fulfilment of the Requirements for the Degree, Doctor of Philosophy. Athens: Georgia.
- Borg, W.R.; and Gall, M.D.(1989). *Educational Research: An introduction*, (Fifth Edition). Longman: London.

Brenner, M., Mayer, R. E., Mosely, B., Brar, T., Duran, R., Reed, B. S., & Webb, D. (1995). Learning by understanding: The role of multiple representations in learning algebra. *American Educational Research Journal*, 34, 663–690.

Briggs, M., & Pritchard, A. (2002). *Using ICT in primary mathematics teaching*. Exeter: Learning Matters.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht, The Netherlands: Kluwer.

Bryman, A. (2008). *Social research methods* (3rd ed.). Oxford, NY: Oxford University.

Burns, R. B. (2000). *Introduction to Research methods*. London: Sage Publications.

Burns, N. and Grove, S.K. (2003). Understanding nursing research, 3rd Edition. Philadelphia. W.B Saunders Company
Chiu, M.M. (2000) Group problem solving processes: Social interactions and individual actions. *Journal for the Theory of Social Behaviour*, 10 (1), 27 – 50 and 600 – 631.

Chiu, M.M. (2008). Flowing toward correct contributions during groups' mathematics problem solving: A statistical discourse analysis. *Journal of the Learning Sciences*, 17(3), 415-463.

Chrysanthou, I. (2008). *The use of ICT in Primary Mathematics in Cyprus: The case of GeoGebra*. A thesis submitted in satisfaction of the requirements for the degree of Masters of Philosophy in Education (International Perspectives in Mathematics Education). Cambridge: University of Cambridge.

Cikla, O. A. (2004). *The effects of multiple representations-based instruction on seventh grade students' algebra performance, attitude toward mathematics, and representational preference*. (Doctor of Philosophy in Secondary Science and Mathematics Education, The Graduate School of Natural and Applied Science of Middle East Technical University). Accessed 09/06/2013 from <http://etd.lib.metu.edu.tr/upload/12605615/index.pdf>.

Clark-Wilson, A. (2010). *Developing ICT for teaching and learning in the mathematics classroom: Learning from the past... and moving forwards...* NAMA 36 Presentation March 2010.

Clements, D.H., (2000). From exercises and tasks to problems and projects - Unique contributions of computers to innovative mathematics education. *The Journal of Mathematical Behavior*, 19 (1), 9-47.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education*, 23(1), 2-33.

Cobb, P., Yackel, E., & Wood, T. (2002). A Constructivist Approach to Second Grade Mathematics: Radical Constructivism in Mathematics Education. *Mathematics Education Library*, 7, 157-176.

Cobb, P. (1994). *Constructivism in mathematics and science education*. Educational Researcher 23: 4.

Cobb, P. (1995). Mathematical learning and small-group interaction: Four case studies. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* 25–129. Hillsdale, NJ: Erlbaum.

Cohen, E.G. (1994). Restructuring the classroom: Conditions for productive small groups. *Review of Educational Research*, 64(1), 1-35.

Cohen, L.; Manion, L. and Morrison, K. (2003). *Research Methods in Education*. 5th Edition. London: RoutledgeFalmer.

Cole, M. (1996). *Cultural Psychology: a once and future discipline*. Cambridge, Ma: Harvard University Press.

Confrey, J. (1990). What constructivism implies for teaching. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist Views on the Teaching and Learning of Mathematics*, *Journal for Research in Mathematics Education Monograph*, 4, 107-122.

Cooper, H. M., (1984). The integrative research review: A systematic approach. *Applied social research methods series* (Vol. 2). Beverly Hills, CA: Sage.

Cordova, D. I., & Lepper, M. R. (1996). Intrinsic Motivation and the Process of Learning: Beneficial Effects of Contextualization, Personalization and Choice. *Journal of Educational Psychology*, 88(4), 715-730.

Cox, M., Abbott, C., Webb, M (2003). *ICT and Pedagogy: A review of the research literature*. Coventry/London: Becta / DfES

Crawford, K. (1996). Vygotskian approaches in human development in the information era. *Educational Studies in Mathematics* , 31(1-2), 43-62. Kluwer Academic Publishers

Creswell, J.W. (2008). *Educational Research: Planning, conducting and evaluating quantitative and qualitative research*. 3rd Edition. New Jersey: Pearson Education, Inc.

Crook, C. (1994). *Computers and the collaborative experience of learning*. London: Routledge.

Crotty, M. (1998). *The Foundations of Social Research: Meaning and perspective in research process*. London: SAGE Publications.

Davis, N. (2001). *The virtual community of teachers*. In M. Leask (Ed.), *Issues in teaching using ICT*. London; New York: RoutledgeFalmer.

Denscombe, M. (2007) *The Good Research Guide for Small-scale Social Research Projects*, 3rd Edition. Buckingham: Open University Press.

de Villiers, M. and Jugmohan, J. (2012). Learners' conceptualization of the sine function during an introductory activity using Sketchpad at Grade 10 Level: *Research paper presented by Janeeshla Jugmohan at the Educational Society of South Africa (EASA) Congress*, Univ. of KwaZulu-Natal.

Dick, T. P. (1992). Super calculators: implications for the calculus curriculum, instruction and assessment. In J. T. Fey & C. R. Hirsch (Eds.), *Calculators in Mathematics Education*, 145–157. Reston, VA: NCTM.

Dikovic, L.(2009). Applications GeoGebra into Teaching Some Topics of Mathematics at the College Level. *ComSIS* 6 (2), 191-203.

Dillenbourg, P. (1999) Collaborative learning: Cognitive and Computational Approaches. *Advances in Learning and Instruction Series*. New York: Elsevier Science Inc.

Drijvers, P., Kieran, C., & Mariotti, M. A. (2010). Integrating technology into mathematics education: Theoretical perspectives. In C. Hoyles & J.-B. Lagrange (Eds.), *Mathematics education and technology -rethinking the terrain*, 89–132. New York: Springer.

Driscoll, M. (1999). *Fostering Algebraic Thinking: A Guide for Teachers Grades 6-10*. Portsmouth, NH: Heinemann.

Dufour-Janvier, B.; Bednarz, N and Belanger, M. (1987). Pedagogical considerations concerning the problem of representation. In C. Janvier (Ed.) *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 109-122).Lawrence Erlbaum Associates: Hillsade, New Jersey.

Duncan, A. G. (2010) Teachers' views on dynamically linked multiple representations, pedagogical practices and students' understanding of mathematics using TI-Nspire in Scottish secondary schools, *ZDM Mathematics Education* 42:763-774. DOI 10.1007/s11858-010-0273-6

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics, *Educational Studies in Mathematics*, (61), 103 – 131.

Engeström, Y. (1999). *Perspectives on activity theory*. Cambridge: Cambridge University Press.

Erickson, F. (1992). The interface between ethnography and microanalysis. In: M. D. LeCompte, W. L. Millroy, & J. Preissle (Eds.), *The handbook of qualitative research in education* (pp. 201–225). San Diego, CA: Academic Press.

Fey, J.T. (1989). Technology and mathematics education: A survey of recent developments and important problems, *Educational Studies in Mathematics*, 20, 237-272.

Falcade, R.; Laborde C. and Mariotti, M. A. (2007). Approaching functions: Cabri tools as instruments of semiotic mediation. *Educational Studies in Mathematics* 66:317–333. DOI 10.1007/s10649-006-9072.

Ferrara, F.; Pratt, D and Robutti, O. (2006). The role and uses of technologies for the teaching of algebra and calculus. In A. Gutierrez, P Boero (Eds.) *Handbook of research on the Psychology of mathematics education: Past, present and future*, 237 – 273. Sense Publishers.

Flick, U. (2006). *An Introduction to Qualitative Research*, 3rd Edition. London: Sage.

Fraenkel, J.R.& Wallen, N.E. (2003). *How to design and evaluate research in education*, (5th ed). Boston: McGraw-Hill.

Gamoran, A.; Secada, W. G.; & Marrett, C. B. (1998). The organizational context of teaching and learning: Changing theoretical perspectives in Hallinan. *Handbook of sociology of education*.

Gawlick, T. (2002). *Restructuring Dynamic Constructions: Activities to Stimulate the Development of Higher Level Geometric Thinking*. Accessed online: Url: www.icme-organisers.dk/tsg10/articulos/Gawlick. [Viewed on 08/01/2010].

Gilakjani, A. P.; Leong, L, and Ismail, H. N. (2013). Teachers' Use of Technology and Constructivism. In I.J. Modern, *Education and Computer Science*, 2013, 4, 49-63. Accessed Online 07/04/2013. Url: <http://www.mecs-press.org>.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: strategies for qualitative research*. New York: Aldine.

Goldenberg, E. P. (1987). Believing is seeing: How preconceptions influence the perceptions of graphs. In J. C. Bergerson, N. Herscovics, & C. Kieran, (Eds.),

Proceedings of the eleventh international conference for the Psychology of Mathematics Education, 1, 197-203. Montreal, Quebec: Program Committee.

Goldin, G. A., & Janvier, C. (1998). Representations and the psychology of mathematics education. *Journal of Mathematical Behavior*, 17(1), 1- 4.

Goos, M.; Galbraith, P.; Renshaw, P. and Geiger, V. (2001). Integrating technology in mathematics learning: What some students say. In J. Bobis, B. Perry & M. Mitchelmore (Eds.), *Numeracy and beyond (Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia)*, 225-232. Sydney: MERGA.

Goos, M.; Galbraith, P.; Renshaw, P. & Geiger, V. (2004) *Perspectives on technology mediated learning in secondary school mathematics classrooms*. Sydney: MERGA.

Gray, D.E. (2009). *Doing Research in the Real World*, 2nd Edition. London: Sage.

Guin, D. & Trouche, L.(1999). The complex process of converting tool into mathematical instruments: the case of calculators, *International Journal of computer for mathematical learning*, 3 (3), 195-227.

Gutierrez, A.; Laborde, C.; Noss, R. and Rakov, S. (1999). Tools and technology in didactics in mathematics. In I. Schwank (Ed). *European Research in Mathematics Education*, 183-188.

Güven, B., & Kosa, T. (2008). The Effect of Dynamic Geometry Software on Student Mathematics Teachers' Spatial Visualization Skills. *The Turkish Online Journal of Educational Technology – TOJET*, 7 (4), Article 11.

Hadas, N.; Hershkowitz, R.; and Schwarz, B. B. (2000). The Role of Contradiction and Uncertainty in Promoting the Need to Prove in Dynamic Geometry Environments: *Educational Studies in Mathematics*, 44, (1/2), 127-150 [Online URL: <http://www.jstor.org/stable/3483207>. Accessed: 07/08/2014 07:42].

Hammersley, M. (1990). *Reading Ethnographic Research: a Critical Guide*, London: Longmans.

Hannafin, R.B; Burruss, J.D.; and Little, C. (2001). Learning with Dynamic Geometry Programmes: Perspectives of Teachers and learners. *The Journal for Educational Research*, 94, 132-144.

Hedberg, J.G. (2007). *Searching for disruptive pedagogies: matching pedagogies to technologies*, 5(12). [Online: accessed on 11/06/2012], URL <http://www.curriculum.edu.au>]

Herceg, D. and Herceg D. (2009). The definite integral and computers. *The teaching of mathematics*, 12 (1), 33-44.

Hiebert, J. and T. Carpenter. (1992). Learning and teaching with understanding. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*, 65-97. New York: Macmillan.

Hoffman, E. S. (2001). Technology planning and implementation: A study of effective change efforts in Michigan public school districts. *Dissertation Abstracts International* 62-05.

Hohenwarter, M. (2004). Bidirectional Dynamic Geometry and Algebra with GeoGebra. *Proceedings of the German Society of Mathematics Education's annual conference on Mathematics teaching and Technology*. Soest, Germany.

Hohenwarter, M. and Fuchs, K. (2005). *Combination of dynamic geometry, algebra and calculus in the software system GeoGebra*. Retrieved from International GeoGebraWiki <http://www.geogebra.org/en/wiki/index.php/publications>.

Hohenwarter, M. (2006). Dynamic investigation of functions using GeoGebra. *Proceedings of Dresden International Symposium on Technology and its Integration into Mathematics Education*. Dresden, Germany: DES-TIME.

Hohenwarter, M.; Preiner, J. (2007): Dynamic Mathematics with GeoGebra. *Journal for Online Mathematics and its Applications*, 7, Article ID 1448.

Hohenwarter, M.; Preiner, J., Taeil Yi, (2007): Incorporating GeoGebra into Teaching Mathematics at the College Level, *Proceedings of ICTCM*, Boston, MA, Accessed online 10/09/2014, Url: http://www.geogebra.org/publications/2007_ICTCM_geogebra.

Hohenwarter, M., & Lavicza, Z. (2007). Mathematics teacher development with ICT: towards an International GeoGebra Institute. In D. Kuchemann (Ed.), *Proceedings of the British Society for Research into Learning Mathematics*. 27(3). University of Northampton, UK: BSRLM.

Hohenwarter, M., & Jones, K. (2007). Ways of linking geometry and algebra: the case of GeoGebra, *Proceedings of British Society for Research into Learning Mathematics*, 27 (3).

Hohenwarter, M., Hohenwarter, J., Kreis, Y., Lavicza, Z. (2008). Teaching and learning calculus with free dynamic mathematics software GeoGebra. *11th International Congress on Mathematical Education*. Monterrey, Nuevo Leon, Mexico.

Hohenwarter, J., & Hohenwarter, M. (2009). Introducing Dynamic Mathematics Software to Secondary School Teachers: The Case of GeoGebra, *Journal of Computers in Mathematics and Science Teaching*, 28 (2), 135-146.

Hollebrands, K. F. (2007). The role of a dynamic software program for geometry in the strategies high school mathematics students employ. *Journal for Research in Mathematics Education*, 38(2):164 — 192.

Holloway, I. and Wheeler, S. (2010). *Qualitative Research in Nursing and Healthcare*, 3rd Edition. Wiley-Blackwell

Hopkins, P. L., (1992). Simulating Hamlet in the Classroom. *System Dynamics Review*, 8 (1), 91-98.

Hershkowitz, R., Schwarz, B. B., & Dreyfus, T. (2001). Abstraction in contexts: Epistemic actions. *Journal for Research in Mathematics Education*, 32(2), 195-222.

Hoyles C. and Jones K. (1998). Proof in dynamic geometry contexts. *In: Perspectives on the Teaching of Geometry for the 21st century-An ICMI study*, C. Mammana and V. Villani (eds.) (chap. 4, section II, 121-128) (Dordrecht: Kluwer Academic Publisher).

Hoyles, C. and Noss, R. (2003). What can digital technologies take from and bring to research in mathematics education? In Bishop, A. J. Clements, M. A., Keitel, C., Kilpatrick J. and Leung, F. K. S (Eds.), *Second International Handbook of Mathematics Education*, (Dordrecht, The Netherlands: Kluwer), 1, 323-350.

Hoyles, C., Noss, R., & Kent, P. (2004). On the integration of digital technologies into mathematics classrooms, *International Journal of Computers for Mathematical Learning*, 9(3), 309-326.

Hoyles C., Noss, R (2009). The Technological Mediation of Mathematics and Its Learning. In Nunes, T (ed.) Special Issue, 'Giving Meaning to Mathematical Signs: Psychological, Pedagogical and Cultural Processes'. *Human Development*, 52, (2), 129-147.

Holzl, R. (2001). Using dynamic Geometry software to add contrast to geometric situations – A case study. *International Journal of Computers for Mathematical Learning*, 6, 63–86.
Kluwer Academic Publishers: Netherlands.

Ittigson, R.J. and Zewe, J.G. (2003). Technology in the mathematics classroom. In Tomei, L.A. (Eds.), *Challenges of Teaching with Technology across the Curriculum: Issues and Solutions*. Hershey: Information Science Publishing, 114-133.

Laborde, C. (2001). Integration of technology in the design of geometry tasks with Cabri-Geometry. *International Journal of Computers for Mathematical Learning*, 6, 283-317.

Janvier, C. (1987). Representations and understanding: The notion of function as an example. In C. Janvier (Ed.), *Problems of Representations in the Learning and Teaching of Mathematics*, 67-73. New Jersey: Lawrence Erlbaum Associates.

Jonassen, D.H. (1991). Objectivism versus constructivism: do we need a new philosophical paradigm? *Journal of Educational Research*, 39 (3), 5 – 14.

Jonassen, D.H. (1991a). Context is everything. *Educational Technology* 31 (6), 33-34.

Jones, K. (2000). Providing a foundation for deductive reasoning: Students' interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational studies in Mathematics*, Vol 44(1/2). Proof in Dynamic Geometry Environments. 55 – 85: Springer Stable.

Johnson, B.R. (1997). Examining the validity structure of qualitative research. *Education* 118 (3), 282 – 292.

Kafai, Y.B. and Resnick, M. (1996). *Constructionism in practice: Designing, thinking and learning in a digital world*. Mahwah, NJ: Lawrence Erlbaum Associates.

Kaptelinin, V. (2005). The object of activity: making sense of the sense-maker. *Mind, Culture and Activity*, 12(1), 4-18.

Kaput, J. J. (1989). Linking representations in the symbol systems of algebra. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra*, 167- 194. Hillsdale, NJ: LEA.

Kaput, J. J. (1992). Technology and mathematics education. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*, 515-556. New York: Macmillan.

Keller, B. A., & Hirsch, C. R. (1998). Student preferences for representations of functions. *International Journal of Mathematical Education and Science and Technology*, 29(1) 1-17.

Keong, C. C.; Horani, S. & Daniel J. (2005). A Study on the Use of ICT in Mathematics Teaching. *Malaysian Online Journal of Instructional Technology (MOJIT)*, 2 (3), 43-51.

Kiczek, R. D. (2000). *Tracing the development of probabilistic thinking: profiles from a longitudinal study*. Rutgers University, New Brunswick.

Kokol-Voljc, V. (2003). Early Development of Mathematical Concepts Using Dynamic Geometry, *The 8th Asian Technology Conference in Mathematics (ATCM 2003)*, Taipei, Taiwan, 202-209.

Kokol-Voljc, V. (2007). Use of mathematical software in pre-service teacher training: The case of DGS. In D Kuchemann (Ed.): *Proceedings of the British Society for Research into Learning Mathematics*, 27 (3).

Kumar, R., 2005, *Research Methodology: a Step by Step guide for Beginners*, Thousand Oak: Sage.

Laborde, C. (2001). Integration of technology in the design of geometry tasks with Cabri-Geometry. *International Journal of Computers for Mathematical Learning*, 6, 283-317.

Lagrange J.B. (1999a). Techniques and concepts in pre-calculus using CAS: a two year classroom experiment with the TI92. *The International Journal for Computer Algebra in Mathematics Education*, 6 (2), 143-165.

Lagrange, J.B. (2008). Ordinary teachers using technology: concerns, theoretical approaches, case studies. *Lecture presented at the 11th International Congress on Mathematical Education*, July 6-13. Monterrey: Mexico.

Lagrange, J.B. (1999). Learning pre-calculus with complex calculators: mediation and instrumental genesis, *Proceedings of the 23rd PME Conference*, Haifa, 3, 193- 200.

Lagrange, J.B. (2008). Ordinary teachers using technology: concerns, theoretical approaches, case studies. *Lecture presented at the 11th International Congress on Mathematical Education*, July 6-13. Monterrey: Mexico.

Lagrange, J.B.; Artigue, M.; Laborde, C. and Trouche, L. (2003). Technology and Mathematics Education: A Multidimensional Study of the evolution of research and Innovation. In A.J. Bishop; M.A. Clements; C. Keitel; J. Kilpatrick; F.K.S. Leung (eds.) *Second International Handbook of Mathematics Education*. Dordrecht: Kluwer Academic Publishers, 237-269.

LaRossa, R. and Reitzes, D.C. (1993). Symbolic Interactionism and Family Studies. *In Sourcebook of Family Theories and Methods: A Contextual Approach*. Eds, P.G. Boss; W.J. Doherty; R. LaRossa; W.R. Schumm and S.K. Steimetz. New York: Plenum.

LeCompte, M. D. and Goets, J. P. (1982). Problems of Reliability and Validity in Ethnographic Research, *Review of Educational Research*, 53: 31-36.

Lehtinen, E.; Hakkarainen, K.; Lipponen, L.; Rahikainen, M. and Muukkonen, H. (2001) *Computer Supported Collaborative Learning: A Review*. [Online Url: <http://www.comlab.hut.fi/opetus/2005/etatehtava1.pdf>. Accessed on 06/06/2012.

Lerman, S. (1996). Intersubjectivity in Mathematics Learning: A Challenge to the Radical Constructivist Paradigm? *Journal for Research in Mathematics Education*, Vol. 27, No. 2 pp. 133-150

Loveless, A. (2003). The interaction between primary teachers' perceptions of ICT and their pedagogy. *Education and Information Technologies*, 8, 313-326.

Lowrie, T. (2001). The influence of visual representations on mathematical problem solving and numeracy performance. In B. Perry (Ed.), *Numeracy and Beyond* (Vol. 2). Sydney: MERGA.

Lingguo, B.; Selcuk, H. E. (2010). Sliders in Dynamic Mathematics Learning Environments: Their Pedagogical Roles, *Mathematics and Computer Education*, 44, (3).

Maher, C. A., & Martino, A. M. (1996a). The development of the idea of mathematical proof: a 5-year case study. *Journal for Research in Mathematics Education*, 27(2), 194–214.

Maher, C. A. (2002). How students structure their own investigations and educate us: what we've learned from a fourteen year study. In: A. D. Cockburn, & E. Nardi (Eds.), *Proceedings of the Twenty-Sixth Conference of the International Group for the Psychology of Mathematics Education*, 1, (31–46). Norwich, UK: School of Education and Professional Development, University of East Anglia.

Mainali, B. R. and Key, M. B. (2012). *Using dynamic geometry software GeoGebra in developing countries: A case study of impressions of mathematics teachers in Nepal*. Accessed online 12/05/2014. Url: <http://www.cimt.plymouth.ac.uk/journal/mainali.pdf>.

Marrades, R. and Gutierrez, A. (2000). Proofs produced by secondary school participants learning geometry in a dynamic computer environment. *Educational Studies in Mathematics*, 44 (1 & 2), 87-125.

Mariotti, M.A. (2002). Introduction to Proof: The mediation of dynamic software environment. *Educational Studies in Mathematics*, 44 (1 & 2), 25-53.

Maynard, M. (1994). Methods, practice and epistemology: the debate about feminism and research. In Mary Maynard & Jane Purvis (Eds.), *Researching women's lives from a feminist perspective*, 10-27. London: Taylor and Francis.

McClintock, E., Jiang, Z., and July, R. (2002). Students' development of three-dimensional visualization in The Geometer's Sketchpad environment. *ERIC Document Reproduction Service No. ED 471759*.

McCracken, G.D. (1988). *The Long Interview*. London: Sage.

McFarlane, A.E., et al., (1995). 'Developing an understanding of the meaning of line graphs in primary science investigations, using portable computers and data logging.' *Journal of computers in mathematics and science teaching*, 14 (4), 461 - 480.

Mehanovic, S. (2011). The Potential and Challenges of the Use of Dynamic Software in Upper Secondary Mathematics Students' and Teachers' Work with Integrals in GeoGebra Based Environments. *Linköping Studies in Science and Technology*. Institute of Technology: Linköping Universitet. Thesis, No. 1499

Mehdiyev, R. (2009). Exploring participants' learning experiences when using a Dynamic Geometry Software (DGS) tool in a geometry class at a secondary school in Azerbaijan. Thesis submitted for the MSc in Mathematics and Science Education. AMSTEL Institute Universiteit Amsterdam: The Netherlands December 24, 2009

Merriam, S. (1998). *Qualitative research and case study applications in education*. San Francisco: Josey-Bass.

Mertens, D.M. (2010). *Research and evaluation in education and psychology: Integrating diversity with quantitative, qualitative, and mixed methods*. (3rd ed.) Thousand Oaks, CA: Sage.

Miles, M.B. and Huberman, A.M. (1994). *Qualitative Data Analysis*. 2nd ed. Sage. London.

Miller, D.J., Glover, D., Averis, D. (2008). Enabling enhanced mathematics teaching: Final Re-port for the National Centre for Excellence in the Teaching of Mathematics. Accessed online 03/09/2013, <http://www.keele.ac.uk/.../keeleuniversity/.../interactivewhiteboard/ncetmreport-1.pdf>

Misfeldt, M. (2008) Semiotic Instruments: Considering technology and representations as complementary. Accessed Online 12/04/2013: Url: <http://archive.geogebra.org/static/publications/2008-Misfeldt-Cerme6.pdf>

Misfeldt, M. (2013). Instrumental Genesis in GeoGebra Based Board Game Design. In B. Ubuz, C. Haser, & M. A. Mariotti (Eds.), *Papers for the CERME 8 Conference* 2664-2673). Ankara, Turkey: Middle East Technical University.

Mitnik, R.; Recabarren, M.; Nussbaum, M.; and Soto, A. (2009). Collaborative Robotic Instruction: A Graph Teaching Experience. *Computers and Education Journal*, 53(2), 330-342.

Monaghan, L. (2004). Doorwork and Legal Risk: Observations from an Embodied Ethnography, *Social Legal Studies*, 13(4), 453-480.

Monaghan, J. and Ozmantar, M.F. (2006). Abstraction and Consolidation, *Educational Studies in Mathematics* 62: 233–258. Springer 2006

National Council of Teachers of Mathematics (NCTM). (2000). Principles and standards for school mathematics. Reston, VA: Author.

Nardi, B. A. (1996). Studying context: a comparison of Activity Theory, Situated Action Models, and Distributed Cognition. Nardi, B. A. (ed.), *Context and Consciousness*, pp. 69-102. Cambridge MA: The MIT press.

Neill, H. and Quadling, D. (2003) *Pure Mathematics 1 & 2: Advanced Level Mathematics*. Cambridge University Press.

Newby, T. J.; Stepich, D. A.; Lehman, J. D. and Russell, J. D. (1996). *Instructional technology for teaching and learning: designing instruction, integrating computers, and using media*, NJ: Prentice Hall.

Noss, R., Hoyle, C. & Kent, P. (2004). On the integration of digital technologies into mathematics classrooms. In : *International Journal of Computers for Mathematical Learning*, 9 (3), 309-326 .

Ofsted (2004). ICT in schools – the impact of government initiatives, Secondary Mathematics London, *A report from the Office of Her Majesty's Chief Inspector of Schools* <http://www.ofsted.gov.uk/assets/3646.pdf>

Ofsted (2008). Mathematics: Understanding the score. Accessed Online 05/06/2012, Url: www.ofsted.gov.uk/resources/070063.

Ofsted (2012) Mathematics: Made to Measure. Accessed Online 30/07/2012, Url: www.ofsted.gov.uk/resources/110159.

Oldknow, A. and Taylor R (2003). *Teaching Mathematics using Information and Communication Technology*. London: Continuum

Onwuegbuzie, A.J. and Leech, N.L. (2007). Sampling designs in qualitative research: making the sampling process more public, *The Qualitative Report*, 12(2): 238-54

- Orton, A. (1983a). Students' understanding of integration. *Educational Studies in Mathematics*, 14(1), 1-18.
- Orton, A. (1983b). Students' understanding of differentiation. *Educational studies in mathematics*, 14(3), 235-250.
- Ozgun-Koca, S. A. (2001). Computer-based representations in mathematics classrooms: The effects of multiple-linked and semi-linked representations on students' learning of linear relationship. Published PhD dissertation. Ohio: Ohio State University.
- Panasuk, R. (2010). Three-phase ranking framework for assessing conceptual understanding in algebra using multiple representations, *EDUCATION*, 131 (4).
- Parahoo, Kader (1997). *Nursing Research: Principles, Process and Issues*. Palgrave.
- Pea, R. D. (1987). Cognitive technologies for mathematics education. In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 89-122). Hillsdale, N.J.; London: Erlbaum.
- Philliber, S. G.; Schwab, M. and Sloss G. S. (1980). *Social Research*. New York: Peacock Publishers, Inc.
- Piaget, J. (1973). *Main Trends in Psychology*. London: George Allen & Unwin.
- Pitts, V. R. (2003). *Representations of functions: An examination of pre-service mathematics teachers' knowledge of translations between algebraic and graphical representations*. Pittsburg: University of Pittsburg
- Polit, D.E. (2001). *Essentials of qualitative research*. 3rd Edition. New York. Lippincott. Williams and Wilkins.
- Poppe, P.E. (1993). Representations of Function and the Roles of the Variable. Dissertation Abstracts International, 54-02, 0464.
- Porzio, D. T. (1994). The Effects of Differing Technological Approaches to Calculus on Students' Use and Understanding of Multiple Representations When Solving Problems. PhD. The Ohio State University.
- Postlethwaite, T. (2005). Educational research: Some basic concepts and terminology. In: Ross K. (ed.) *Quantitative research methods in educational planning*. Paris, International Institute for Educational Planning. UNESCO.
- Powell, A.B.; Francisco, J.M.; and Maher, C.A, (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *Journal of Mathematical Behaviour*, 22, 405-435.

- Preiner, J. (2008). Introducing dynamic mathematics software to mathematics teachers: the case of GeoGebra. Dissertation in mathematics education. Faculty of natural sciences. University of Salzburg.
- Rabardel, P. (1995). People and technology – a cognitive approach to contemporary instruments: in N. Sinclair, (2001). *The aesthetic is relevant for the learning of mathematics*. 21(1), 25 – 33.
- Rajasekar S.; Philominathan P.; Chinnathambi V. (2006) Research Methodology: Accessed online: url: arXiv:physics/0601009[physics.ed-ph]. Date: 07/06/2013
- Ridgeway, J. (1997). Vygotsky, informatics capability and professional development. In: Passey, D. & Samways, B. (eds.) *Information technology supporting change through teacher education*. London: Chapman Hall.
- Robson, C. (2009). *Real World Research*, 2nd Edition. Carlton, Victoria, Australia: Blackwell Publishers.
- Rogoff, B. (1990) *Apprenticeship in Thinking: Cognitive Development in Social Context*, Oxford University Press, Oxford, UK.
- Roschelle, J.R. (2000). Choosing and using video equipment for data collection. In R. Lesh & A. Kelly (Eds.), *Handbook of research design in mathematics and science education*, pp (457 – 486). Mahwah NJ: Lawrence Erlbaum Associates Inc.
- Ruthven, K. (1990). The influence of graphic calculator use on translation from graphic to symbolic forms. *Educational Studies in Mathematics* 21(5), 431-450. JSTOR
- Ruthven, K., & Hennessy, S. (2002). A practitioner model of the use of computer-based tools and resources to support mathematics teaching and learning. *Educational Studies in Mathematics*, 49(1), 47-88.
- Ruthven, K., Hennessey, S. (2003) Successful ICT use in secondary mathematics-a teacher perspective. *Micro Math* 19 (92) 20
- Ruthven, K.; Deaney, R.; Hennessey, S. (2009). Using graphing software to teach about algebraic forms: a study of technology supported practice in secondary school mathematics, *Educational Studies in Mathematics*, 71 (3) 279-293
- Ruthven, K.; Hennessy, S.; and Deaney, R. (2008). Constructions of dynamic geometry: A study of the Interpretative flexibility of Educational Software in Classroom Practice. *Computers in Education Journal*, 51 (1) pp. 297-331.

TTA. (1999). Using ICT to meet teaching objectives in art - ITT primary. Accessed online 07/06/2014, Url: <http://www.canteach.gov.uk/community/ict/exemplification/index.htm>

Sarankatos, S. (2005). *Social Research (3rd Edition)*. Palgrave MacMillan, China.

Saunders, M., Lewis, P., and Thornhill, A. (2007). *Research methods for business students (4th edition)* Harlow: Pearson Education.

Schoenfeld, A.H.; Smith, J.P. and Arcavi, A. (1993). Learning: The micro-genetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (ed.). *Advances in instructional psychology*. Volume 4 pp 55 – 175.

Schwartz, J., and Yerushalmy, M. (1992). Getting students to function in and with algebra. In E. Dubinsky & G. Harel (Eds.) *The concept of function: Aspects of epistemology and pedagogy (pp. 261–289)*. Washington DC: MAA.

Selinger, M.; Pratt, D. (1997). Mediation of mathematical meaning through the graphic calculator, *Journal of Information Technology for Teacher Education*, 6:1, 37-50. Accessed online, 27/09/2013. Url: <http://dx.doi.org/10.1080/14759399700200003>.

Serhan, D. (2009). Using concept maps to assess the effect of graphing calculators use on students concept images of the derivative at a point. *International journal for mathematics teaching and learning*, October, 116-210.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22(1), 3–22.

Sharpe, R., Benfield, G., Lessner, E., & DeCicco, E. (2005). Final report: Scoping study for the pedagogy strand of the JISC learning programme, http://www.jisc.ac.uk/uploaded_documents/scopingstudyfinalreportv4.1.doc [28/7/14].

Silverman, D. (2011). *Doing qualitative research*, 3rd ed. London: Sage.

Simon, M, A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26(2), 114-145.

Simpson, M., Tuson, J. A., (2003). *Using observations in small-scale research: a beginner's guide* (Rev. ed.). Glasgow: Scottish Council for Research in Education.

Skemp, R. R. (1987). *The psychology of Learning Mathematics*. Hillsdale, NJ: Lawrence Erlbaum Associations.

Smart, T (1995) Visualisation, confidence and magic; the role of graphic calculators. In L. Burton & B. Jaworski (Eds) *Technology in Mathematics Teaching: a bridge between teaching and learning*. London: Chartwell-Bratt.

Smith, J.A.; Flowers, P. and Larkin, M. (2012). *Interpretive Phenomenological Analysis: Theory, Method and Research*. Sage: London.

Smith, B.L.; and MacGregor, J.T. (1992). *What is Collaborative Learning?* National Centre on Post-Secondary Teaching, Learning and Assessment at Pennsylvania State University.

Steffe. L.; and Gale. J. (Eds.) (1995). *Constructivism in education*. Hillsdale. NJ: Lawrence Erlbaum.

Straesser, R. (2001). Cabri-Geometry: Does Dynamic Geometry Software (DGS) Change Geometry and its teaching and learning. *International Journal of Computers for Mathematical Learning*, 6, 319 -333.

Swafford, J. O. & Langrall. C. W. (2000). Grade 6 students' pre-instructional use of equations to describe and represent problem situations. *Journal for Research in Mathematics Education*, 31(1), 89-112.

Tall, D. (1985). Using computer graphics programs as generic organisers for the concept image of differentiation. In L. Streefland (Ed.), *Proceedings of the 9th PME International Conference*, 1, 105–110.

Tall, D. (1991). The psychology of advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking*, 3–21. Dordrecht, The Netherlands: Kluwer.

Tall, D. (1996). Function and calculus. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *International handbook of mathematics education*, 1, 289–325. Dordrecht, The Netherlands: Kluwer.

Tam, M. (2000). Constructivism, Instructional design and technology: Implications for transforming distance learning. *Educational Technology & Society* 3(2) 2000. [accessed online 02/08/2014. Url. http://www.ifets.info/journals/3_2/tam.html

Templer, R., Klug, D., & and Gould, I. (1998). Mathematics laboratories for Science graduates. In C. Hoyles, C. Morgan & G. Woodhouse (Eds.), *Rethinking the mathematics curriculum*, 140-154. London: Farmer Press.

Terhart, E. (2003). Constructivism and teaching: a new paradigm in general didactics? *Journal of Curriculum Studies*, 35(1), 25 - 44

Trouche, L. (2005). Instrumental genesis, individual and social aspects. In D. Guin, K. Ruthven & L. Trouche (Eds.), *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument* (pp. 198-230). New York, NY: Springer.

Van Voorst, C. (1999) Technology in mathematics teacher education. Online: Accessed on 13/03/2013. Url: http://www.ictc.org/T99_Library/T99_54.PDF

Verillion, P. & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education*, 9(3), 77 - 101.

Vergnaud, G. (1996). The Theory of Conceptual Fields. In L. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of Mathematical Learning*, 219-240. Mahwah, NJ: Lawrence Erlbaum Associates.

Von Glasersfeld, E. (1989). Cognition, Construction of knowledge and technology: Synthese 80, 121 – 140. Online: Accessed on 12/04/2013. Url: <http://www.univie.ac.at/constructivism/EvG/papers/118.pdf>

von Glasersfeld, E. (1995). A constructivist approach to teaching. In L. Steffe & J. Gale (Eds.) *Constructivism in education*, 3-16. Hillsdale, NJ: Lawrence Erlbaum.

Vrasidas, C. (2000). Constructivism versus objectivism: Implications for interaction, course design, and evaluation in distance education. *International Journal of Educational Telecommunications*, 6(4), 339-362.

Vygotsky, L. S. (1978). *Mind in society. The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Wertsch, J.V (1991). *Voices of the mind: A sociocultural approach to mediated action*. Harvard University Press. Cambridge: Mass, USA.

Wertsch, J. V.; & Addison Stone, C. (1985). The concept of internalization in Vygotsky's account of the genesis of higher mental functions. In J. V. Wertsch (Ed.), *Culture, communication and cognition: Vygotskian perspectives*, 162–166. New York: Cambridge University Press.

White, T. (2008). Debugging an artifact, instrumenting a bug: Dialectics of instrumentation and design in technology-rich learning environments. *International Journal of Computers for Mathematical Learning*, 13(1), 1-26.

Wilensky, U. (1991). Abstract Mediation on the Concrete and the Concrete Implications for mathematics Education. In Harel G. & S. Papert (eds.) *Constructivism*. Norwood, NJ: Ablex Publishing Corporation.

Yerushalmy, M. (1991). Student perceptions of aspects of algebraic function using multiple representation software. *Journal of Computer Assisted Learning*, 7, 42-57.

Yerushalmy, M. (1997). Reaching the unreachables: Technology and semantics of asymptotes. *International Journal of Computer for mathematical Learning* 2: 1-25.

Yerushalmy, M., Schwartz, J.L. (1999) A procedural approach to exploration in calculus. *International Journal of Mathematical Education in Science and Technology*, 30 (6), 903-914.

Yıldırım, A. Şimşek, H.(2006). *The Qualitative Research Methods in Social Sciences*, Ankara: Seçkin

Yin, R. K. (2009). *Applications of case study research (4th ed.)*. Thousand Oaks, CA: Sage.

APPENDIX A

Sketch the following graphs

1. $y = |x + 3|$

2. $y = |3x - 1|$

3. $y = |x - 5|$

4. $y = |3 - 2x|$

5. $y = -2|2x - 1|$

6. $y = 3|2 - 3x|$

7. $y = |x + 4| + |3 - x|$

8. $y = |6 - x| + |1 + x|$

9. $y = |x^2 - 2|$

10. $y = 2|x - 1| - |2x + 3|$

Draw sketches of each of the following set of graphs

1. $y = \sin(x)$ and $y = |\sin(x)|$ and $y = \sin(|x|)$

2. $y = (x - 1)(x - 2)(x - 3)$ and $y = |(x - 1)(x - 2)(x - 3)|$ and $y = (|x| - 1)(|x| - 2)(|x| - 3)$

[Source: Neill, H. and Quadling, D. (2003)]

APPENDIX B: Modulus Function Worksheet

Solve the following equations.

1. $|2x| = 7$
2. $|x| + 10 = 0$
3. $|x + 6| = 7$
4. $2|x - 5| + 1 = 3$
5. $|x - a| = 1$
6. $|x + 2| = |x + 3|$
7. $|3x - 2| = |7x - 1|$
8. $x + 2 = |x| + |2x - 1|$
9. $|x - 2| + |x + 2| = |x^2| + 7$
10. $x = |-x| + |x| - |2x - 3|$.
11. $7 - |x + 3| = 2|x - 2| + |x - 1|$.
12. $|x + 1| - |x| + 3|x - 1| - 2|x - 2| = x + 2$

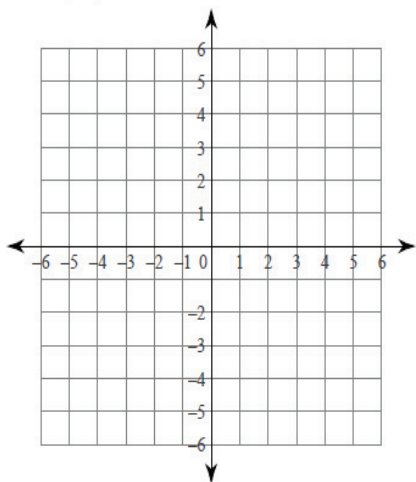
Sketch the following functions.

1. $y = |x - 3|$.
2. $y = |2x + 1|$.
3. $y = |x - 4| - 4$.
4. $y = |x| - |x - 2|$.
5. $y = |x| + |x - 2|$.
6. $y = |x| - |x + 3| + |3x - 1|$.
7. $y = |x^2 - 4|$.
8. $y = |x^2 - 4| - |x^2 - 1|$.
9. $y = |x^2 - 4| + |x^2 - 1|$.
10. $y = |x^3 - x| - |x^3|$.

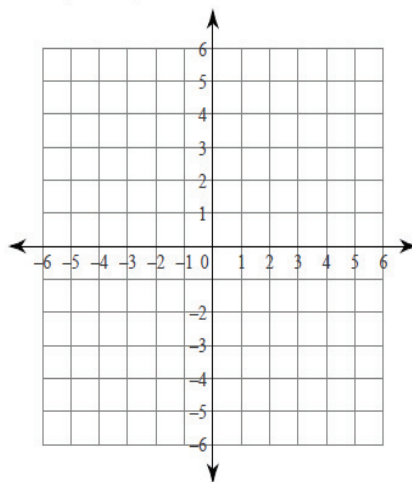
APPENDIX C

Graph each equation.

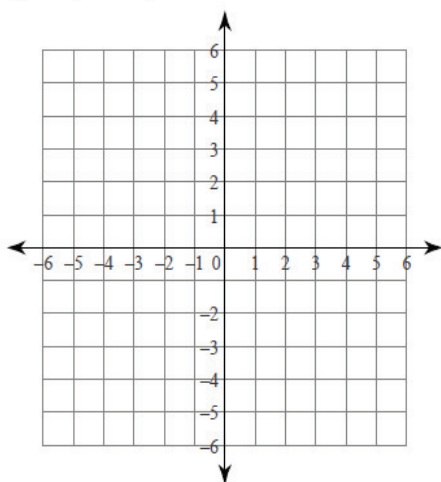
1) $y = |x| + 1$



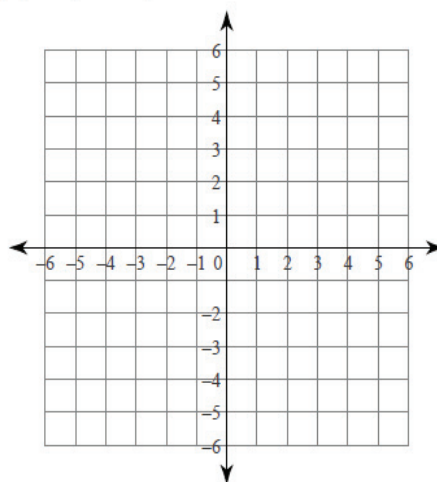
2) $y = |x - 2|$



3) $y = |x + 3| - 4$

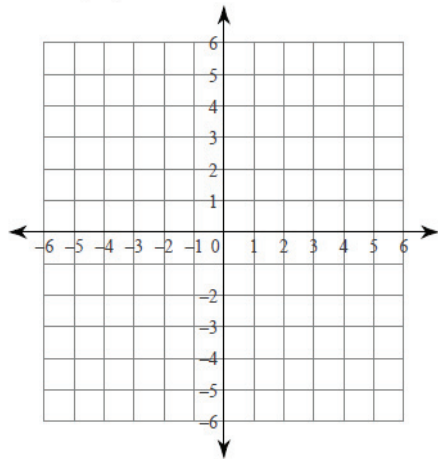


4) $y = |x - 1| + 3$

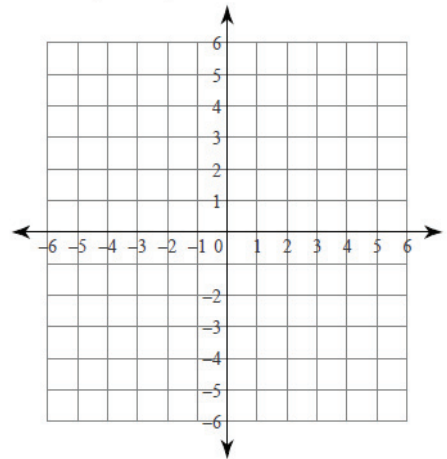


APPENDIX C

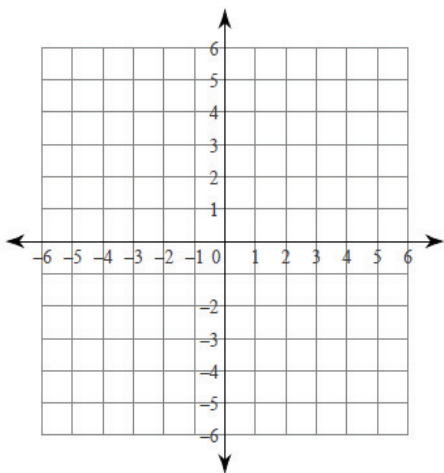
5) $y = -|x| + 2$



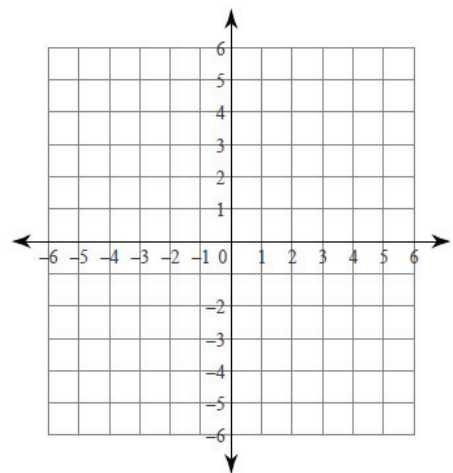
6) $y = -|x - 3|$



7) $y = -|x - 2| + 4$

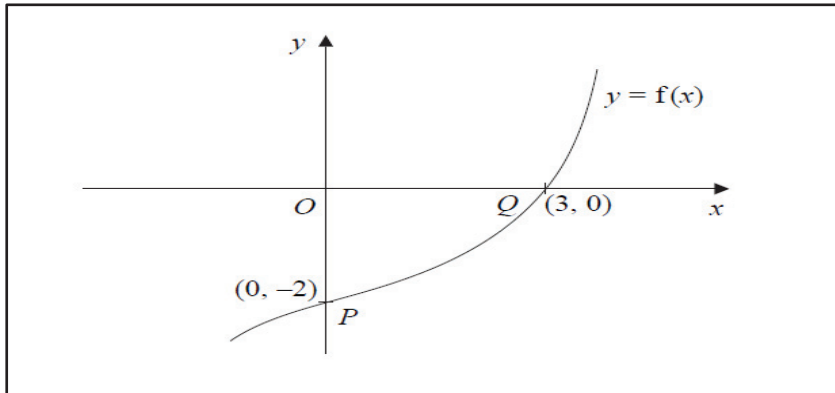


8) $y = -|x + 1| - 2$



APPENDIX D

Question 1:



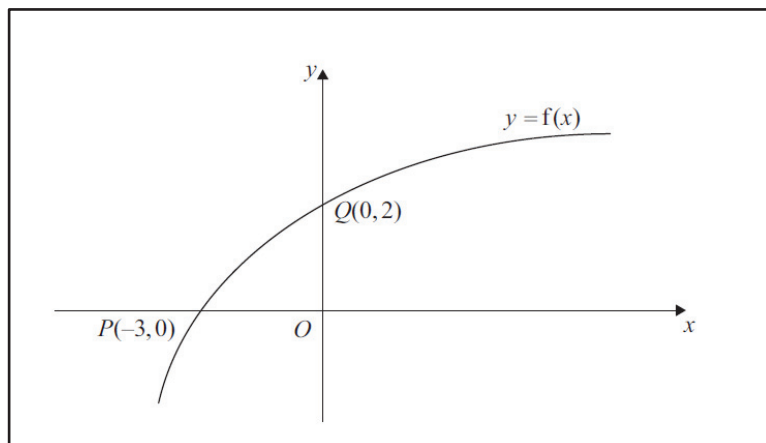
The diagram shows part of the curve with equation $y = f(x)$, $x \in \mathbb{R}$, where f is an increasing function of x . The curve passes through the points P (0, -2) and Q (3, 0) as shown.

In a separate diagram, sketch the curves with equations;

(c) $y = |f(x)|$

(d) $y = f(|x|)$

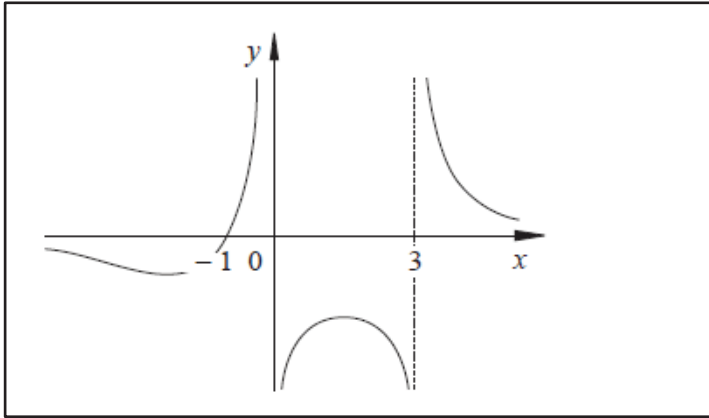
Question 2:



The diagram shows part of the curve with equations $y = f(x)$, $x \in \mathbb{R}$. The curve passes through the points Q(0, 2) and P (-3, 0) as shown. On separate diagrams, sketch the curve with equation $y = f(|x|) - 2$, (Jan 2013 Qu. 3).

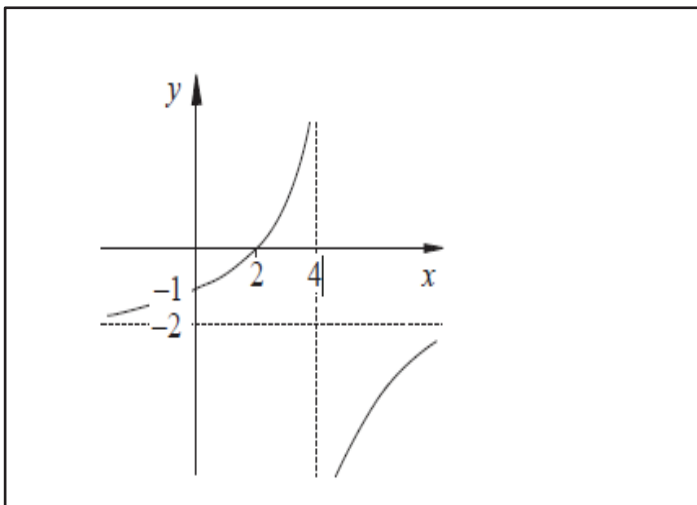
APPENDIX D

Question 3:



The diagram shows the graph of $y = f(x)$. On separate diagrams, sketch the graphs of $y = |f(x)|$ and $y = f(|x|)$.

Question 4



The diagram shows the graph of $y = f(x)$. On separate diagrams, sketch the graph of $y = |f(x)|$ and the graph of $y = f(|x|)$.

Question 5:

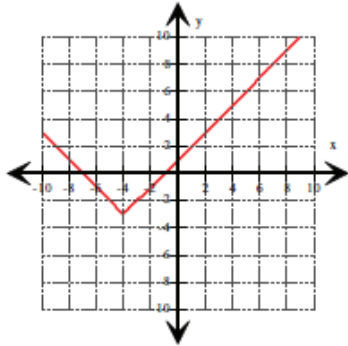
The function f is defined by $f: \rightarrow \ln(2x - 1)$ and g is defined by $g: \rightarrow \frac{2}{x-3}$.

- Sketch the graph of $y = |g(x)|$. Indicate clearly the equation of the vertical asymptote and the coordinates of the points where the graph crosses the y-axis
- On the same grid sketch the graph of $y = f(x)$
- Estimate the value of $\left| \frac{2}{x-3} \right| = \ln(2x-1)$

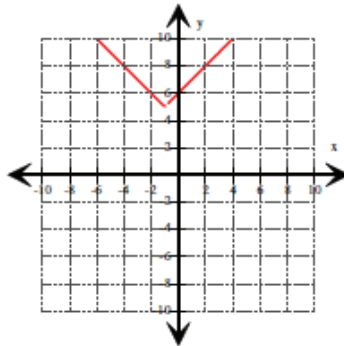
APPENDIX E

A graph of absolute value function is shown in each question. Find the function.

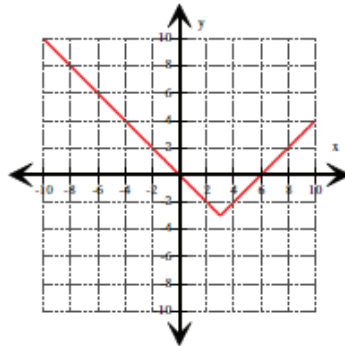
1) $f(x) =$



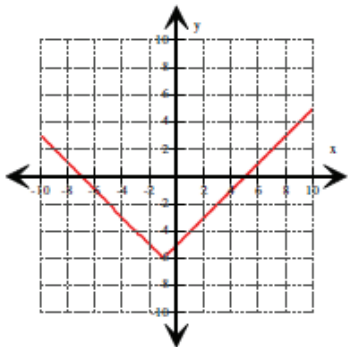
2) $f(x) =$



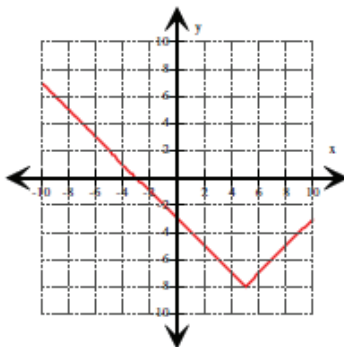
3) $f(x) =$



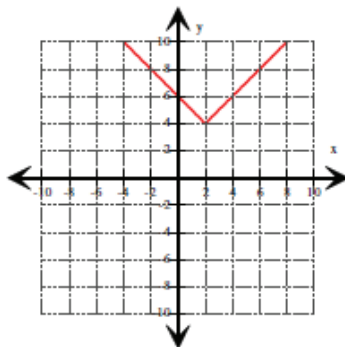
4) $f(x) =$



5) $f(x) =$



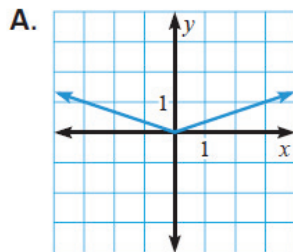
6) $f(x) =$



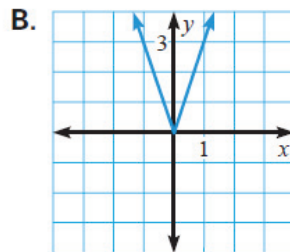
APPENDIX F

EXAMINING THE EFFECT OF a Match the function with its graph.

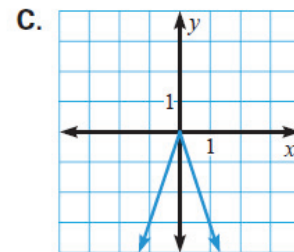
12. $f(x) = 3|x|$



13. $f(x) = -3|x|$

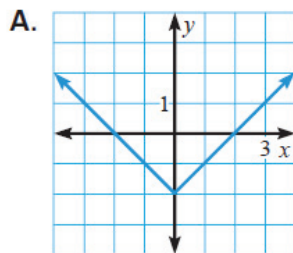


14. $f(x) = \frac{1}{3}|x|$

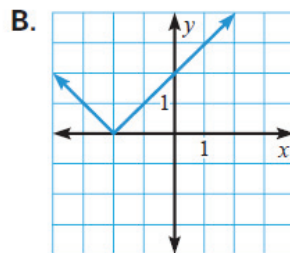


EXAMINING THE EFFECTS OF h AND k Match the function with its graph.

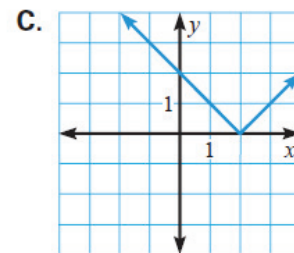
15. $y = |x - 2|$



16. $y = |x| - 2$



17. $y = |x + 2|$



APPENDIX G

LETTER OF CONSENT:

This is a study on the contributions of technology on mathematics learning: Characterising the variety of classroom practices associated with using technology in Post 16, 'A' level (A2) mathematics.

Dear Student,

During the course of this year, April 2014 to October 2014, I am engaged in the process of improving the learning of A-level mathematics using technology as part of my doctoral study with the Institute of Education, London.

The study seeks to investigate the potential contributions of GeoGebra software in the development of your ability to solve mathematical problems at GCE advanced level. This study will explore how you use GeoGebra software to understand mathematical concepts such as modulus functions. It attempts to analyse how multiple representation features of the software can aide in the learning of new mathematical concepts.

Studies have shown that technology can be used in a variety of ways to improve and enhance the learning of mathematics. With the aide of technology, mathematics concepts can be presented in a variety of ways, such as diagrams, schemes, drawings and graphs. The features embedded in technology enables us to see dynamic changes to objects in real time, providing instant feedback, while at the same time allowing us the freedom to solve problems without restrictions of pencil and paper.

The study process involves video and audio recordings of sessions while you are using GeoGebra. The recordings attempt to capture how you will be using GeoGebra and the approaches you will use to solve problems while working with the software. Only your voices and on-screen activities will be video recorded.

This is an on-going case study which involves data collection through the use of interviews and screencast video recordings of some of our sessions while participating in this study. You will work in pairs most of the time and may be required to record yourself solving problems on modulus functions. Your input is greatly valued and I hope the process will help you to understand the concept of modulus functions better.

More details will be provided at the meeting once you consent to participate in the study. Occasionally you will be asked to undertake activities, probably during lunch times or soon after school. You can consider this as akin to attending an after school club.

Most of the activities in this research will be discussed as a group activity and the times negotiated. The activities agreed would fit with your curriculum and will be embedded within the learning process. For those activities that require extra time we will agree on the course of action. This research is a journey we will undertake as a team to effect change to our approach to learning. I am looking forward to it.

Your participation is voluntary. If you choose not to participate or to withdraw from the study at any time, there will be no penalty. It will not affect your grade or affect the help you get from me as a lecturer. You must feel free to express your honest opinion without fear of reprisals. You are free to view the data collected about you at any stage of the research.

The results of the research may be published, but your name will not be used. Any videotapes that are recorded and any written work collected as part of this study will be used only with other educators for research or educational purposes. During the research project, all tapes will be securely kept in my home. The tapes will be erased or destroyed six months after the research project is completed.

If you have any questions concerning the research study, please contact Mr Ebert Gono at High Pavement College (Mathematics Department).

Sincerely,
Ebert Gono
Mathematics Lecturer

I _____, give consent to participate in the above study. I understand that I will be videotaped during class and my written work will be copied as outlined above.

☐ *I agree to have my voice, actions videotaped.*

☐ *I do not agree to have my voice, actions videotaped*

Signature and Date

Guardian/Parent's Signature

APPENDIX H

Edexcel Core 3 (A2) Specifications on Modulus Functions:

The modulus function.	Students should be able to sketch the graphs of $y = ax + b $ and the graphs of $y = f(x) $ and $y = f(x)$, given the graph of $y = f(x)$.
Combinations of the transformations $y = f(x)$ as represented by $y = af(x)$, $y = f(x) + a$, $y = f(x + a)$, $y = f(ax)$.	Students should be able to sketch the graph of, for example, $y = 2f(3x)$, $y = f(-x) + 1$, given the graph of $y = f(x)$ or the graph of, for example, $y = 3 + \sin 2x$, $y = -\cos\left(x + \frac{\pi}{4}\right)$. The graph of $y = f(ax + b)$ will <i>not</i> be required.

AQA Core 3 (A2) Specifications on Modulus Functions.

14.1 Algebra and Functions Definition of a function. Domain and range of a function.	Notation such as $f(x) = x^2 - 4$ may be used. Domain may be expressed as $x > 1$ for example and range may be expressed as $f(x) > -3$ for example.
Composition of functions.	$fg(x) = f(g(x))$
Inverse functions and their graphs.	The notation f^{-1} will be used for the inverse of f . To include reflection in $y = x$.
The modulus function.	To include related graphs and the solution from them of inequalities such as $ x + 2 < 3 x $ using solutions of $ x + 2 = 3 x $.
Combinations of the transformations on the graph of $y = f(x)$ as represented by $y = af(x)$, $y = f(x) + a$, $y = f(x + a)$, $y = f(ax)$.	For example the transformations of: e^x leading to $e^{2x} - 1$; $\ln x$ leading to $2\ln(x - 1)$; $\sec x$ leading to $3\sec 2x$ Transformations on the graphs of functions included in modules Core 1 and Core 2.

Mathematics Education in Industry (MEI): Core 3 (A2) Specifications on modulus functions:

The modulus function:
9 Understand the modulus function.
10 Be able to solve simple inequalities containing a modulus sign.

OCR Core 3 (A2) Maths Specifications on modulus functions:

Algebra and Functions

Candidates should be able to:

- (a) understand the terms function, domain, range, one-one function, inverse function and composition of functions;
- (b) identify the range of a given function in simple cases, and find the composition of two given functions;
- (c) determine whether or not a given function is one-one, and find the inverse of a one-one function in simple cases;
- (d) illustrate in graphical terms the relation between a one-one function and its inverse;
- (e) use and recognise compositions of transformations of graphs, such as the relationship between the graphs of $y = f(x)$ and $y = af(x+b)$, where a and b are constants;
- (f) understand the meaning of $|x|$ and use relations such as $|a| = |b| \Leftrightarrow a^2 = b^2$ and $|x-a| < b \Leftrightarrow a-b < x < a+b$ in the course of solving equations and inequalities;
- (g) understand the relationship between the graphs of $y = f(x)$ and $y = |f(x)|$;
- (h) understand the properties of the exponential and logarithmic functions e^x and $\ln x$ and their graphs, including their relationship as inverse functions;
- (i) understand exponential growth and decay.

APPENDIX I

Date _____

1. Classroom Observation:

A. *Learning Integration*

2. How well participants got onto and begin using GeoGebra for a specific task?
3. How well did participants navigate their own way through specific tools on the GeoGebra menu?
4. Are participants doing the work they were assigned to do? Did they have additional programs/games running simultaneously during lesson time?
5. Do participants seem to be able to complete activities within the allotted time frame? Describe the quality of student work.
6. How often did participants ask me for help?
7. Did participant ask peers for additional help instead of me?
8. Is the learning environment collaborative and encouraging?
9. Did participants link their investigations to previous knowledge?
10. Were there an emerging themes linked to but outside the defines of the curriculum specification

B. *Affective domain and technology aspiration:*

Do participants appear to generally be happy, bored, frustrated, distracted, entertained, etc. during their activities?

C. Guiding questions during group discussions:

1. Can you tell me something about your computer use during the session
2. Did you enjoy using GeoGebra? Why?
3. How much do you feel the use of GeoGebra has helped?
4. Do you feel GeoGebra has improved your understanding of modulus functions and made your learning easier?
5. Is there anything else anyone would like to share?

APPENDIX J

Interview data from Session 6:

1. Can you tell me something about your use of GeoGebra during the activities?

Emma: I used GeoGebra a lot during the first activities to understand modulus functions. I also used it to investigate more complicated functions. It also helped me to consolidate my knowledge of domains and ranges of functions.

2. Did you find it easy to find the correct commands and tools? Which features did you find easy/difficult to use?

James: I found GeoGebra easy to use and I have now downloaded the App on my iPhone. I found it easy to learn how to use GeoGebra, while investigating modulus functions. Initially it was just funny to enter function after function on the screen and I did not even bother to link the graphs and their functions. At first I ended up having a screen full of different graphs, but with the introduction of colour codes, I started to notice the links between functions and their graphs. This encouraged me to investigate more functions and note the dynamic changes taking place in the multiple screens. At first I did not even notice that the algebra screen at times looked different from what I would have typed. E.g. typing $y = \text{abs}(x)$ would appear as $f(x) = |x|$.

It is unfortunate that we are not allowed to use GeoGebra in an examination. It would have made my life easier. However, I now understand the difference between $y = |f(x)|$ and $y = f(|x|)$.

John: It was extremely easy to navigate around GeoGebra. I think it is better than using my graphics calculator in that I can move from one function to another with great ease. This is because in GeoGebra I can use sliders to modify functions in the different windows without the need for me to type new parameter. GeoGebra has clear images and it is interesting to see graphical representations which are easy for me to understand.

James and John's observations were not different from the rest of the participants and they concur with Dikovic (2009) who argues that GeoGebra is more user-friendly compared to a graphic calculator, and that it offers an easy-to-use interface, multilingual menus, commands and a help line.

3. Did you enjoy using GeoGebra? Why?

Sophie: I enjoyed using GeoGebra and I learnt a lot about modulus functions..... Now I understand why at times my algebraic solutions are marked wrong. I now make rough sketches first to provide me with guidance when solving equations and inequalities. The work became more interesting as I worked through more complicated problems. I never knew that I could differentiate and integrate functions using GeoGebra.

4. Is there anything you failed to do on GeoGebra during the activities?

Peter: I managed to do all the tasks given to me. However, there were cases when I failed to understand some graphical representations that appeared on the screen. This was the case with the representations for the sum of two modulus functions and graphs of derivative functions. I took for granted that the given diagrams were correct and just moved on to the next task. I thought it would be better if GeoGebra could include some verbal explanations to the graphical representations.

5. How much do you feel the use of GeoGebra has helped in these sessions?

Sophie: The real time changes that took place in the multiple screens helped me understand the different graphical representations of modulus functions. Without GeoGebra, I think I would have restricted my work to linear functions only. With GeoGebra I managed to investigate graphs involving trig functions, algebraic rational functions, logarithms and exponentials. I also know the difference between the graphs of $y = |f(x)|$ and $y = f(|x|)$.

6. Do you feel GeoGebra has improved your understanding of modulus functions and made your learning easier?

Susan: I got an insight on how to solve equations involving modulus functions. Some of them were complicated but I am now confident that I can answer any examination question on modulus functions. The sessions stretched and challenged my thinking. I managed to investigate aspects beyond the curriculum but all this helped me to understand modulus functions better.

7. Is there anything else anyone would like to share?

James: I prefer to sketch the graph of modulus function first, look for intersection of the lines. If the line is the part of the original graph of the function $y = f(x)$, I remove the modulus sign. If the line is part of $y = f(x)$ reflected in the y-axis, I multiply equation for the line by -1, then solve like a normal inequality.